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GRADED CHAINS OF WORD PROBLEMS OR DON'T BE AFRAID OF PARAMETERS

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Abstract:
In the article, we suggest the necessary change of priorities within mathematics education, i.e. to replace the accent on training of particular computational algorithms with the accent on solver capabilities. This new requirement in mathematics education can also be fulfilled through the parametric tasks. Thanks to the parameters, one can discover various dependencies between values affecting the final result of the task. The parametrization of the problem leads to the creation of solving of the entire system of tasks. The ability to parametrize a task and subsequently to solve the created parametric problem represents considerable benefit and extension of pupils' solver skills.

Keywords: mathematical pattern, parameter, problem solving skills, problem posing, task parametrization, variable

1. Introduction

The word problems leading to equations and inequalities with a parameter represent a traditionally challenging curriculum often causing struggling for learners. Despite of this fact we think that the parametrization can be very useful tool not only for problem solving but also for pupils' and teachers' problem posing as a process of creating meaningful mathematical problems from interpretations established through concrete cases and based on mathematical experiences. Working with parameters represents great didactic potential enabling teachers to teach mathematics with understanding. We assume the concept of a parameter to be introduced as the required part of pupils' problem solving skills. A parameter is used to write the tasks system with dependence of result on later determined value of a parameter. We are interested in the understanding of the way the pupils cope with a parameter, that is, an indeterminate but fixed element of the values taken by a variable.

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The idea of introducing a parameter into a question statement can be shown in the following simple example. The set of solutions of the inequality $x > 9$ can be written as interval $K = (9; \infty)$. The necessity of the introducing a parameter can be justified as follows: We are looking for an answer to the question: "How to carry out the correctness test?" It should be emphasized that also in the case of inequalities it is necessary to have a possibility to verify the correctness of our solution even if the test is not the necessary part of the solution. The numbers satisfying the inequality can be written in two manners – as the inequality $x > 9$ and in the form of the set $K = (9; \infty)$. In both cases it is the set of all real numbers greater than 9. If the analogy with correctness test for equations is used, we should gradually substitute all real numbers greater than 9. But there are infinitely many of these numbers. In this context, it is appropriate to recall for pupils that within the correctness test it is not possible to substitute for the variable neither an inequality nor a set. One can substitute just a number. Therefore we need to find a new way of writing all real numbers greater than 9 and it should have a form of a number. Each real number greater than 9 can be obtained by adding a positive real number to the number 9. Consequently, the real numbers greater than 9 can be written as $9 + a$; $a \in (0; \infty)$. In our case we used the "letter" for the writing and the "letter" is called a parameter. In general, the concept of a parameter is being used in mathematics if we want to write down more objects with a common property using one writing (expression, formula, equation…). The writing $9 + a$ represents the infinite set of all real numbers greater than 9 and substituting this expression to the inequality the correctness test is done by one calculation for infinitely many solutions of the inequality.

We will show particular examples of parameters use for word problems modification and progressive creation so-called graded chains of word problems (Švrček, 2014). Generally, it means a sequence of thematically related and following to each other problems ordered with progressive difficulty. Similar approach to problem solving can be found in work of other authors (Polák, 1996, 1999; Odvárko, 1962; Kopka, 1999; Hejný, 1990). In our case there is a sequence of modifications of the same word problem making by changing or adding some parameters.

A. Task 1

(Šedivý, 1978). How many liters of water must evaporate from a container with 1600 liters of 6% salt solution to remain at least 8% solution?

Solution:

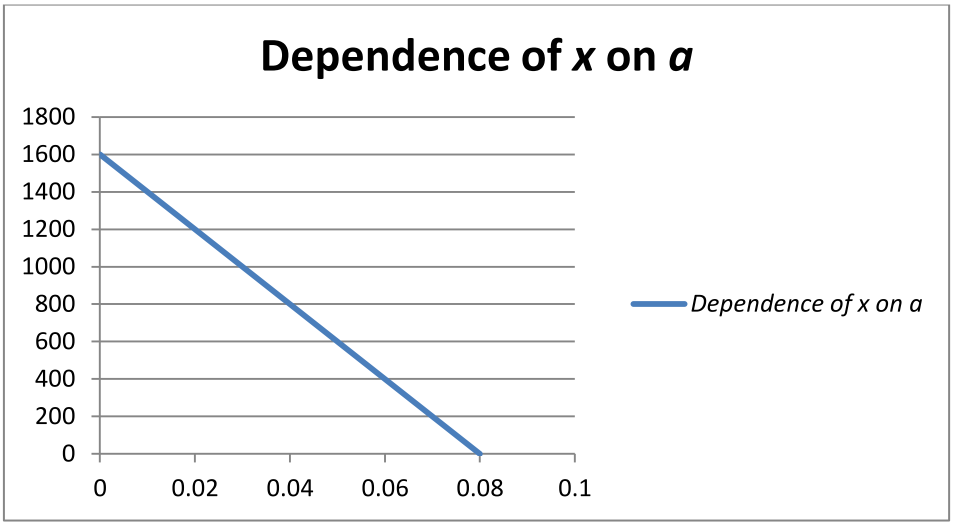
If we denote the number of liters of evaporated water by x, the concentration of the solution after the evaporation is expressed as a fraction $\frac{1600.0{,}06}{1600-x} = \frac{96}{1600-x}$. So we are dealing with inequality $\frac{96}{1600-x} \geq 0{,}08$ and we can easily see that our task suits values $x \geq 400$. This means that if at least 8% of solution should remain, at least 400 liters of water must be evaporated.

Modification 1:

How many liters of water must evaporate from a container with 1600 liters of a % of salt solution to remain at least 8% solution?

Solution:

Using similar idea we come to the conclusion that the inequality $\frac{1600a}{1600-x} \geq 0{,}08$ with a parameter a needs to be solved. Our task then matches the values $x \geq 1600 - 20000a$. This means that if at least 8% of solution should remain, at least 1600 - 20000a liters of water must be evaporated. It is advisable to discuss with pupils a condition for a parameter a under which the task is meaningful. Obviously, we require the fulfilment of the condition $a < 0, 08$, i.e. that the concentration of the solution at the beginning would be less than at the end and so it would be meaningful to talk about evaporation. The situation is depicted on the graph 1. The horizontal axis represents the values of the parameter a (initial percentage of the salt solution) and the vertical axis represents values of x (amount of evaporated water). The points lying on the thick line corresponds to remained concentration exactly 8%. For example, for $a = 0, 06$ we have $x = 400$ and we obtain our origin task. That means we must evaporate exactly 400 liters of water for the initial concentration of the solution a = 6% to get exactly 8% of solution. If we evaporate more than 400 liters the concentration is more than 8% (points above the thick line). If we evaporate less than 400 liters the concentration is less than 8% (points below the thick line). Similarly, x values can be read for other values of the parameter a and a family of tasks can be solved in this way.



Graph 1

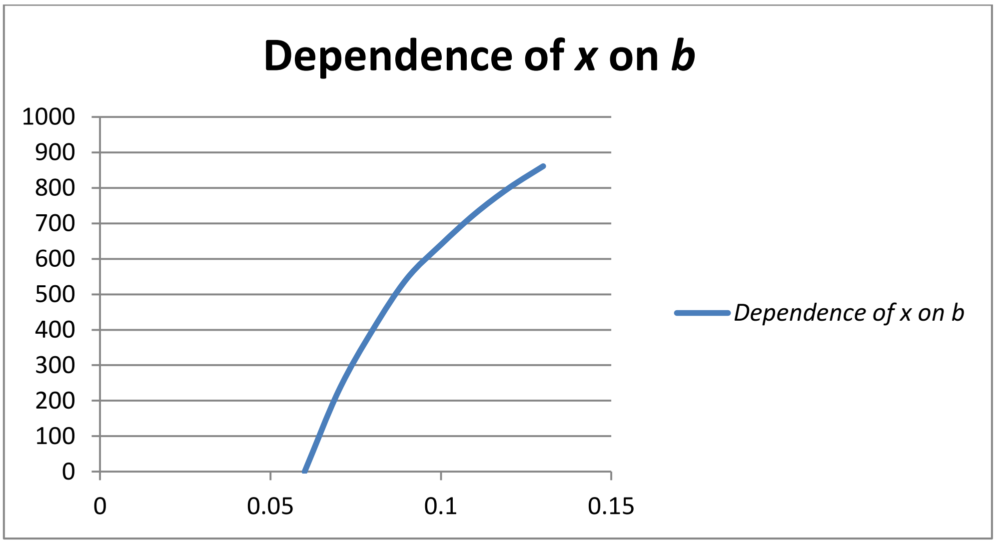
Modification 2:

How many liters of water must evaporate from a container with 1600 liters of 6% salt solution to remain at least b% solution?

Solution:

This time, the solution leads to inequality $\frac{96}{1600-x} \geq b$ with a parameter b. Our task meets the values $x \geq \frac{1600b - 96}{b}$. This means that if at least b% solution should remain, at least $\frac{1600b - 96}{b}$ liters of water must be evaporated.

By setting the specific values for a parameter b, the pupils should conclude that we obtain negative values of x for $b < 0, 06$ which can be interpreted in such a way that the concentration of the solution cannot be reduced by evaporation. A part of the graph of dependence of the amount of evaporated water x on the final concentration of the solution (parameter b) is depicted in the graph 2. The horizontal axis represents the values of the parameter b (final percentage of the salt solution) and the vertical axis represents values of x (amount of evaporated water). The points lying on the thick line corresponds to remained concentration exactly b%. For example, for b = 0, 08 we have x = 400 and we obtain our origin task again.



Graph 2

B. Task 2

A thermometer that can measure temperature in degrees Celsius and Fahrenheit showed 5°C and 41°F at one moment in the morning and 10°C and 50°F at one moment in the afternoon. Using this information:

a) Derive the formula for calculating the values of temperature in degrees Fahrenheit from the values of temperature in degrees Celsius.
b) Derive the formula for calculating the values of temperature in degrees Celsius from the values of temperature in degrees Fahrenheit.
c) Specify the temperature at which the thermometer shows the same value on both scales. Can it happen?

Solution:

a) Since the temperature difference of 9 °F corresponds to the difference of 5 °C, the ratio of the blocks on both scales is 9:5 = 1.8. Thus, one block on the Celsius scale corresponds to 1.8 block on the Fahrenheit scale. Now we can easily express the temperature of 0°C in degrees Fahrenheit. We know that the tempereture of 5°C corresponds to 41°F.

Hence, if the temperature drops by 5 blocks on the Celsius scale, it drops by 5x1.8 = 9 blocks on the Fahrenheit one. That means, the temperature of 0°C corresponds to the temperature of 32°F. One can see that the relation $t_f = 1{,}8t_c + 32$ between the temtperature t_c in degrees Celsius and the temperature t_f in degrees Fahrenheit holds.

b) Now just express $t_c = \frac{t_f - 32}{1{,}8}$.

c) Suppose there exists a temperature at which $t_f = t_c$. Then it would $t_c = 1{,}8t_c + 32$ apply. We get $t_c = -40$ from here. That is, at the temperature of -40°C the thermometer shows the temperature of -40°F as well.

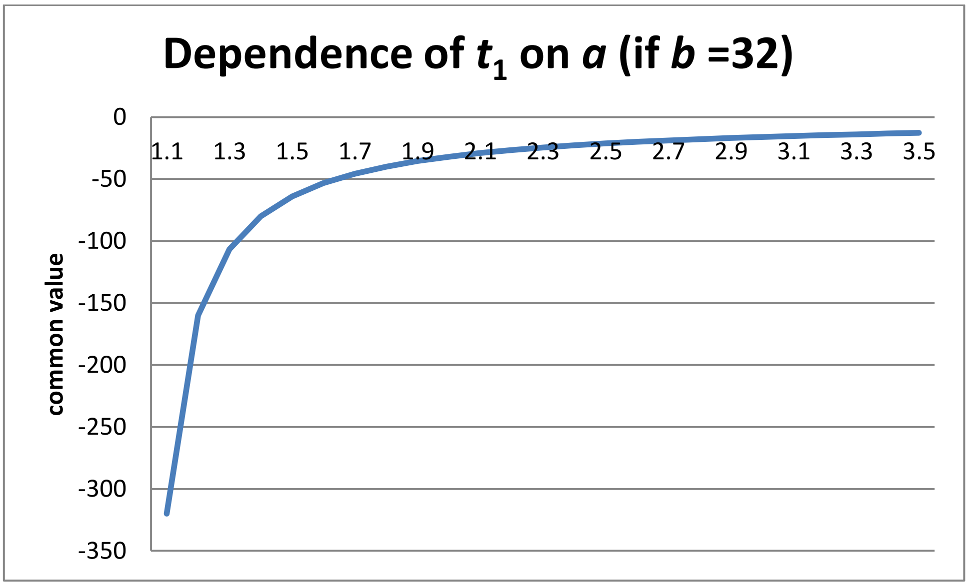
Modification of the task:

Consider two similar scales.

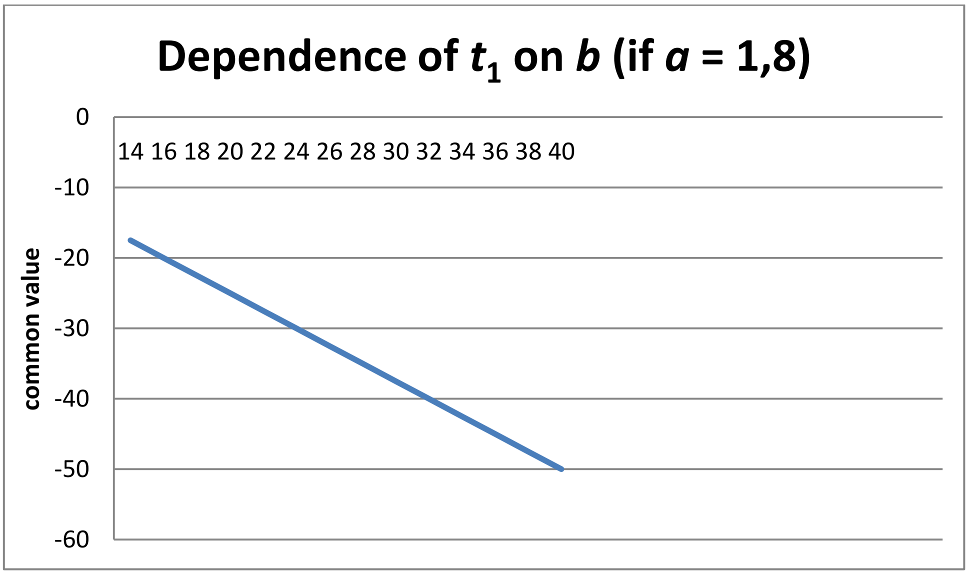
We can introduce two parameters: a – ratio of scales blocks (a = 1,8 in our task), b – zeros offset (b = 32 in our task).

We easily adjust the relationship derived in a) using the parameters. We get a relationship to calculate the temperature on scale 1 using the value on scale 2: $t_1 = at_2 + b$.

Now we can discuss the solution of the point c) with respect to the parameters a and b. Both scales will show the same value if $t_1 = t_2$, i.e. $t_1 = at_1 + b$ and hence $t_1 = \frac{b}{1-a}$. One can see that if the scales do not have a common zero (i.e. $b \neq 0$), the situation where the scales show the same value at the same time can occur only if $a \neq 1$, that is, if the scales do not have the same partitions. We can also discuss when this "common value" of both scales will be negative (as in our task) and when it will be positive. Obviously this value will be negative if $b > 0$ and simultaneously $a > 1$ (our task) or $b < 0$ and $a < 1$. The common value will be positive if $b > 0$ and at the same time $a < 1$ or $b < 0$ and a > 1. The following graphs show the dependence of the "common value" on the ratio of scales blocks a (with fixed zeros offset b = 32) and the dependence of the "common value" on the zeros offset b (with fixed ratio of scales blocks a = 1,8). It can be seen and discussed the dependences using the graphs.



Graph 3



Graph 4

2. Conclusion

The changing requirements of society should be reflected also by objectives of mathematics education. Demands for human numerical skills have decreased with the advent of modern computer technology but at the same time, the requirements for the ability to solve problems have risen. In other words, from today's man is expected that he/she figures out how to solve a given problem and the necessary calculations will be done using a computer. It does not mean that pupils would not learn to compute. We only suggest the necessary change of priorities within mathematics education, i.e. to replace the accent on training of particular computational algorithms with the accent on solver capabilities. This new requirement in mathematics education can also be fulfilled through the parametric tasks. If we use a parameter (or more parameters) during solving a particular problem, it causes a deflection from the calculation of the particular problem to the finding of the solution of the given problem in general. Thanks to the parameters one can discover various dependencies between values affecting the final result of the task. The formula which can be "input" to a computer to make necessary computing after inputting particular values of the parameters is common product of the "parametrization" of the task. The parametrization of the problem leads to the creation of solving of the entire system of tasks. The ability to parametrize of a task and subsequently to solve the created parametric problem represents considerable benefit and extension of pupils' solver skills. Another benefit of the parametric tasks is creating of the correct concept of parameter as a replacement for a number for finding an effective manner of solving given task. Pupils get acquainted the crucial difference between the unknown which should be calculated and the parameter whose values are (will be) determined by particular task assignment. Last but not least, the graded chains of problems bring a new look at the already created (not only mathematical) formulas.

Mathematics education enriched by the graded chains of problems associated with the parameterization offers pupils the ability to solve a particular problem with general insight. Due to the parametrization pupils realize that a particular equation is a mathematical model of given particular situation but the corresponding parametric equation represents a model of entire system of problems. We think that this knowledge is a great asset, for example, for programmers. Their job is often to create a complex task solution (the best formula creation) which is a core of given program. The computer retrieves the required values (parameters) from the submitter of specific requirements and the desired result is available in a short time. This is an example when human has created a way of a solution and subsequently the man has taught the computer to compute the particular tasks from the given task system. This fulfils the society's demand for mathematics education: human creates – computer computes.

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