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PARAMETRIZATION AS AN ASSISTENT IN PHYSICAL PROBLEM SOLVING

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Abstract:
The article deals with physical problems solving and derivation of general physical formulas using parameterization. There is shown in particular examples how the physical problems can be generalized by introducing suitable parameters which then allows to effectively solve the entire series of tasks. The authors also draw attention to the possibilities of didactic use of this procedure within mathematics and physics education, especially the usefulness of discussion on the problem solution according to the parameters values. Last but not least, the parameterization of the physical problems contributes to better learners′ understanding of relationships between physical quantities.

Keywords: mathematics and physics education, parameterization, problem solving, physical formula

Introduction

Mathematics is a science that helps other disciplines to find answers to questions and solve problems. In the conditions of school education, pupils very often encounter using mathematics to solve physical problems. From a mathematical point of view, these are word problems for whose solving the already created formulas are used, similarly like for example in geometry. These formulas express dependencies between individual physical quantities. The right choice of physical formula assumes knowledge of the conditions under which the formula applies. For example, one of the most known formulas $s = v.t$ expresses the dependence between path, velocity and time in uniform motion. Under other conditions, this dependency does not valid, so this formula cannot be used, for example, in the case of accelerated motion. As soon as solving a physical problem the correct formula is chosen, mathematization of the task occurs and that is why mathematics works helpfully in solving the problem. The role of mathematics

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education is to develop the ability of pupils to use mathematics in everyday life - to solve word problems. The great benefit of teaching mathematics for its application in solving physical problems is to enrich pupils to correctly understand physical formulas and to work effectively with them while solving a physical task. In the following examples, we will point out the transition from mathematical tasks to physical ones, with an emphasis on correctly perceiving physical formulas. The solutions are based on the gradual replacement of the specific numerical values with the parameters, or else on the parameterization of the task. This procedure can be used to derive the general physical formulas. The problem of a proper understanding of the concept of the parameter by pupils was dealt with by a number of researchers (Šedivý, J. (1976, Furinghetti & Paola, 1994, Bardini, Radford & Sabena, 2005, Postelnicu & Postelnicu, 2016), who draw attention to the difficulty of introducing this concept. On the contrary, we would like to show that the appropriate implementation of the parameters can make solving of some problems easier. We will show that it is advantageous to look at the physical formulas as relationships with the parameters. The parameterization process enables us to efficiently solve the entire series of tasks at the same time, instead of dealing with specific cases.

Task 1. An owner of a restaurant has two kinds of coffee. One kilogram of the first coffee costs € 5 and one kilogram of the second coffee 14 €. These two kinds of coffee were mixed using 4kg cheaper and 9kg more expensive coffee. What is the resulting coffee price per kilogram?

Solution: It is a standard problem about mixtures. The price of one kilogram of the mixed cofee is:

$$\frac{4.5 + 9.14}{4 + 9} = 11.23$$

The price of the created mixture of coffee is € 11.23.

However, our goal is not fulfilled by finding the solution. This example, which should not be very difficult for pupils, is a reflection point for continuing the "story" about the restaurant owner. The owner is an intriguing person and is trying to prepare a variety of news for his customers. If he blends the coffee in a different ratio, the resulting coffee will have another taste. He could also mix several kinds of coffee, for example, five kinds of coffee with different flavors and different prices. It is interesting for him to mix the coffee so that the resulting coffee has a fixed price. At the same time, he would like to make several different mixtures with different prices, from the original coffee types. These questions can be answered individually, that is, for each ratio of coffee blending to make the previous calculation. Here, of course, the natural question arises as to whether it is possible to create a formula to calculate the required data. Its creation could make the calculations more efficient. But what is more important, the formula could be entered into a computer. The calculations would then be performed by the computer according to the input data. To create a formula, it is necessary to replace specific values with the appropriate parameters. We are talking about task parameterization. Instead of a specific task, we solve the system of tasks to which the particular task belongs. In the first phase of the parameterization, the amount of coffee used is replaced. Let's use l kg cheaper and d kg more expensive coffee to make the mix. Then for the price c of one kilogram of the mixture made from the two coffee we have:

$$c = \frac{l.5 + d.14}{l + d} \quad . \tag{1}$$

This formula contains two kinds of "letters". The parameters l, d, for which we assign numerical values according to a particular task, and then calculate the value of the unknown c. The created formula can be modified to the following form:

$$\frac{l}{d} = \frac{14 - c}{c - 5} \quad . \tag{2}$$

This shape of the formula draws attention to the fact that the resulting price is determined by the mixing ratio of the individual kinds of coffee. One can see that there are several possible combinations of the quantities of the coffee types. This shape of the formula is therefore suitable for calculating the amount of individual kinds of coffee in the mixture for the given final price. At the same time there is an obvious interval of possible blend prices. Simultaneously, the look at the "letters" in the formula changes. The parameter is c, the value of which will be determined and then will be calculated the unknowns l and d. For example, if we want a mix with a price of 7 € per kilogram, so we obtain:

$$\frac{l}{d} = \frac{14 - 7}{7 - 5} = \frac{7}{2} \quad .$$

So, if we mix 7kg cheaper and 2kg more expensive coffee, the final price of one kilogram will be 7 €. The parameterization can continue and it can be created a formula for the final price of a coffee mix if the prices of individual coffee types are not constant. So for any two kinds of coffee. We will replace the price of the cheaper coffee with the parameter k_l and the price of the more expensive coffee with the parameter k_d. Then formula (1) is changed to:

$$c = \frac{l.k_l + d.k_d}{k_l + k_d} \quad . \tag{3}$$

In the formula (2), the "letters" exchanged roles to the formula (1). The parameter became unknown and vice versa. It can be said that what is unknown and what parameter is determined by the specific input when using the formula. In principle, the unknown is the quantity in the formula that we have to calculate under the task assignment. The other "letters" in the formula can be perceived in most cases as parameters. This change of the "letters" in the formulas is an important knowledge when using physical formulas, for example. For physical reasoning is also interesting a dimensional analysis, which serves as one of the basic tests of the correctness of the created formula. It consists of substituting the basic units for the quantities contained in the formula. The same basic unit has to be made on both sides of the formula by subsequent "calculation". Let's make the dimensional analysis for formula (3):

$$c = \frac{l.k_l + d.k_d}{l + d} = \frac{kg.€ + kg.€}{kg + kg} = \frac{kg.€}{kg} = € \,.$$

From the dimensional analysis it follows that according to our formula the price will be calculated in €, which corresponds to reality. Within the mathematics lessons, there is more time to address the mathematical nature of the physical tasks than in the physics ones. We also recommend to spend time deriving already-known formulas used to solve physical problems. Creating a formula will contribute to a better understanding of the numerical values and the units assigned to the individual physical quantities. An interesting way of deriving physical formulas at the secondary school level, which is based on the unit structure of the formulas, is described in the article by Jozef Kvasnica. (Kvasnica, 1991)

Task 2. A cyclist moves at an average speed of 28 km / h. How long does it take 168 km?

Solution: To solve this example, physical formulas are not required. The key to solving the example is the specified speed of the cyclist. The average velocity can be perceived as a constant speed throughout the movement. The numerical value in conjunction with the speed unit indicates the path of the cyclist per unit of time, in our case in one hour. In order to create a mathematical model, that is, an equation, it is necessary to realize that the traveled distance is directly proportional to the driving time. Thus, the given example can be solved using the well-known rule of three.

1 hour .... 28 km
t hours................................ 168 km

Thus, we obtain:

$$t = \frac{168}{28} = 6 \; hours.$$

We have solved one specific example that is a part of a system of problems about uniform motion. Creation of a formula will make it possible to respond more effectively to various questions related to the uniform motion problems. The replacement of the specific numerical data by parameters leads to the creation of the formula. The universal task should have the following assignment:

A cyclist moves at an average speed of v km / h. How long does it take s km?

Solving of this task has the same idea structure as in the case with the specific numbers.

The entry of the rule of three would look as follows:

1 hour .... v km
t hours................................ s km

Hence:

$$t = \frac{s}{v} \ .$$

After adjusting, we get $s = v.t$, which is the well-known formula for the path of the uniform motion. If pupils are involved in creating the formula, they also have the opportunity to see what's behind it. At the same time, the formula creation contributes to a better understanding of the given formula and thus to its more efficient use. In our case, pupils can see that the formula $s = v.t$ is a "product" of mathematical-physical considerations. Its use makes the finding a solution of the problem more efficient, because the considerations and calculations that are in fact directed to the given formula are no longer needed.

In this way, pupils promote their awareness of the need to formulate the formulas as a comprehensive solution to the entire system of tasks. It should be noted that the formula s = v.t is not only a formula for calculating of the path for a uniform motion. It is appropriate for students to learn them to perceive the formulas as a dependency between the values (quantities) occurring in the given formula. The value which we calculate is then perceived as a variable and the values that are specified are perceived as parameters. Another important point for the management of physical movement tasks is necessary to realize that if there are more independently moving objects in the task, the motion of each of them is described with a "self" formula. Based on the assignment, it is necessary to determine which parameters in the "self" formulas acquire the same numerical values. It is also possible to construct "additional" equations by the analysis of the input. The result of the task assignment analysis is a system of equations. Here are two typical examples of the mutual movement of two objects.

Task 3. A car went to a city 62 km distant. It was moving at an average speed of 45 km/h. The car should pick up a passenger in the city and return at the same speed back. In order to save time, the passenger went across the car at an average speed of 6 km/h, so the car reached him in front of the city. How long did the car go there and back?

Solution: It is appropriate to divide the example into two parts. The first part deals with the mutual movement of the car and the passenger opposite one another. The car movement is described by the equation $s_A = 45.t_A$ and the movement of the passenger by the equation $s_P = 6.t_P$. From the mathematical point of view, we have two equations with four unknowns. Based on the above, we consider whether some of the unknowns do not always have the same numerical values. As the car and the passengers set off against each other at the same time, the same time they move to the meeting point. Therefore, $t_A = t_P = t$ is satisfied. Further, it is clear that $s_A + s_P = 62$ holds true. After these considerations, we solve a system of equations with three unknowns. After inserting the right sides of the first two equations into the third equation, we get:

$$45t + 6t = 62 \,.$$

Therefore, the duration of the movement of the car to the passenger $t = \frac{62}{45+6} =$ 1.2 hours. The second part of the solving is focused on the movement of the car after the passenger arrives. It is evident that on the way back the car passes the same path as it has gone to the meeting with the passenger. As the car also returns at an average speed of 45 km / h, the back journey takes the same time. Therefore, the total time of the car is 2.4 hours. In common life, it is often problematic to explain (define) a well-known concept or to justify a "clear" argument. Therefore, it is an interesting question of "proof" that the way back for the car lasts as long as the way to the meeting place. And this is the reason for continuing to think about the given example. We warn pupils that the evidence with specific values is insufficient. It has only a verification function, if the claim for a value is valid and therefore it is important to seek a general proof. The way to the general proof leads through the replacement of specific values by parameters – through the task parameterization. For the time t needed to pass the car to the meeting point (analogous to the specific calculation) we have $t = \frac{s}{v_A+v_P}$, where s is the distance of the starting position of the car from the city. The return path s_s is the original distance between the car and the passenger shortened by the path passed by the passenger at time t. Hence:

$$s_s = s - v_P.t = s - \frac{v_P.s}{v_A + v_P} \,.$$

Then for the time t_s needed for the return of the car is valid:

$$t_s = \frac{s - \frac{v_P.s}{v_A + v_P}}{v_A} = \frac{s}{v_A + v_P} \,.$$

So, $t = t_s$ applies for any initial distance between the car and the passenger and for their any speeds. This proves the validity of the claim in general. At the same time, our general relationship for time highlights the well-known fact that when two solids move in opposite of each other, their speeds add up. What can be interpreted in such a way that the time of the meeting is the same as if one solid was standing and the other was approaching at a speed equal to the sum of the speeds of both solids. There is also offered a discussion of the consequences of the head-on collision of two vehicles as their crash speeds are also adding.

Task 4. Peter Sagan in a stand-alone escape on the rise at Passo dello Stelvio with 24.3 km long has a head start of 15 minutes at a time when the peloton arrives at the foot of the hill. The followers move at an average speed of 16 km / h. Will Sagan win if he moves at an average speed of 14 km / h?

Solution: The assignment of the task differs slightly from the standard example of "catching" two solids. In order to make the most of the knowledge and skills we have already used to solve this example, we transform the task into a standard type. By simple calculation, we find that Sagan goes 3.5 km in 15 minutes. At that point, the peloton will also rise. Therefore, instead of the given example, we solve an example: Sagan is 3.5 km before the peloton. Both the peloton and Sagan moved at the specified speeds at the same time. Will the peloton catch Sagan up during 24.3 km? The motion of the peloton is described by the equation $s_P = 16. t_P$ and the motion of Sagan is described by the equation $s_S = 14. t_S$. The transformed assignment implies that at the point of catch-up of Sagan by the peloton $t_p = t_S$. At the same time, $s_P = s_S + 3.5$ holds. After substituting into the last equation, we obtain:

$$16. t_P = 14. t_S + 3.5.$$

And from there, the time needed to Sagan's catch-up is 1.75 hours. At that time, the peloton would have passed the path $s_p = 28\ km$, which is more than the length of the hill. Therefore, the peloton will not catch Sagan up before the goal of the race, so Sagan will manage to win. After solving the task, natural questions are offered, such as: At least what speed should the peloton go to reach the Sagan before the goal? What was the minimum speed enough for Sagan to win? What minimal lead should Sagan have to win? To be able to answer these and similar questions without repeating all the calculations, it is a good idea to solve the problem in the parametric way, which in practice means creating a "formula" for a given system of tasks. Creating the formula we will work with the parameters: speed of the peloton v_p, speed of Sagan v_S , Sagan's lead time in hours t_n. Sagan's motion is described by the equation $s_S = v_S. t$ and the motion of the peloton is described by the equation $s_P = v_P. t$ (both move the same time). For the peloton's path is valid:

$$s_p = s_S + s_n$$

and after substituting:

$$v_p. t = v_S. t + v_S. t_n\ .$$

After the adjustment, we get the formula for calculation the time needed to Sagan's catch-up:

$$t = \frac{v_S. t_n}{v_p - v_S}\ .$$

After substituting time t into the formula describing the motion of the peloton, we obtain the formula for calculating the path that the peloton needs to reach Sagan:

$$s_p = \frac{v_p . v_S}{v_p - v_S} . t_n \ .$$

Thanks to this formula, the above questions can be answered quickly and easily without recurring calculations. For example, if we want to know the maximum possible Sagan's lead for the peloton to catch him up with it, just substitute the appropriate speed values and for the path s_p the length of the rise. For this task, one can skip the length of the rise in the assignment. Pupils will discover that this information is missing and they will try to get it from the internet while finding pictures of legendary rise and beautiful nature.

Task 5. A thermometer that can measure temperature in degrees Celsius and Fahrenheit showed 5°C and 41°F at one moment in the morning and 10°C and 50°F a tone moment in the afternoon. Using this information

a) Derive the formula for calculating the values of temperature in degrees Fahrenheit from the values of temperature in degrees Celsius.
b) Derive the formula for calculating the values of temperature in degrees Celsius from the values of temperature in degrees Fahrenheit.
c) Specify the temperature at which the thermometer shows the same value on both scales. Can it happen?

Solution: a) Since the temperature difference of 9 °F corresponds to the difference of 5 °C, the ratio of the blocks on both scales is 9:5 = 1.8. Thus, one block on the Celsius scale corresponds to 1.8 block on the Fahrenheit scale. Now we can easily express the temperature of 0°C in degrees Fahrenheit. We know that the tempereture of 5°C corresponds to 41°F. Hence, if the temperature drops by 5 blocks on the Celsius scale, it drops by 5x1.8 = 9 blocks on the Fahrenheit one. That means, the temperature of 0°C corresponds to the temperature of 32°F. One can see that the relation $t_f = 1{,}8t_c + 32$ between the temtperature t_c in degrees Celsius and the temperature t_f in degrees Fahrenheit holds.

b) b) Now just express $t_c = \frac{t_f - 32}{1{,}8}$.
c) Suppose there exists a temperature at which $t_f = t_c$. Then it would $t_c = 1{,}8t_c + 32$ apply. We get $t_c = -40$ from here. That is, at the temperature of -40°C the thermometer shows the temperature of -40°F as well.

Modification of the task:

Consider two similar scales.

We can introduce two parameters: a – ratio of scale blocks ($a = 1.8$ in our task), b – zero offset ($b = 32$ in our task).

We easily adjust the relationship derived in a) using the parameters. We get a relationship to calculate the temperature on scale 1 using the value on scale 2:

$$t_1 = at_2 + b.$$

Now we can discuss the solution of the point c) with respect to the parameters a and b. Both scales will show the same value if $t_1 = t_2$, i.e. $t_1 = at_1 + b$ and hence:

$$t_1 = \frac{b}{1-a}.$$

One can see that if the scales do not have a common zero (i.e. $b \neq 0$), the situation where the scales show the same value at the same time can occur only if $a \neq 1$, that is, if the scales do not have the same partitions. We can also discuss when this common value of both scales will be negative (as in our task) and when it will be positive. Obviously this value will be negative if $b > 0$ and simultaneously $a > 1$ (our task) or $b < 0$ and $a < 1$. The common value will be positive if $b > 0$ and at the same time $a < 1$ or $b < 0$ and $a > 1$.

Task 6. (Polya, 1962) An iron ball is floating in mercury. Water is poured over the mercury and covers the ball. Will the ball sink, rise, or remain at same depth?

Solution: Well-known Archimedes' principle allows the buoyant force of an object partially or fully immersed in a liquid to be calculated. In the first case (before the pouring the water), the gravitational force and the buoyancy force of the mercury act on the ball. In the second case (after the pouring the water), the buoyancy force of the water affects the ball above that. To solve the problem it is enough to consider just volumes, densities and consequently weights of the investigated solids and fluids. The weight of the ball is balanced by the combined weights of the two displaced fluids (mercury and air in the first case and mercury and water in the second case). Let us denote by V the total volume of the ball, by V_u the volume of the part of the ball that is in the upper fluid (air, water) and by V_l the volume of the part of the ball that is in the lower fluid (mercury). Similarly, let ρ be the density of the ball, ρ_u the density of the upper fluid and ρ_l the density of the lower fluid. Then we have $V_u\rho_u + V_l\rho_l = V\rho$. Solve for V_u and V_l in terms of ρ, ρ_u, ρ_l and V:

$$V_u = \frac{\rho_l-\rho}{\rho_l-\rho_u}V,\ V_l = \frac{\rho-\rho_u}{\rho_l-\rho_u}V.$$

Further, we take the following approximate density values (regardless of units): air – $\rho_u = 0$ (the first case), water – $\rho_u = 1$ (the second case), iron – $\rho = 8$ (both cases) and mercury – $\rho_l = 14$ (both cases). Ten we get in the first case $V_u = \frac{6}{14}V,\ V_l = \frac{8}{14}V$ and in the second case we obtain $V_u = \frac{6}{13}V$ and $V_l = \frac{7}{13}V$. Since the volume V_u of the part of the ball above the mercury is greater in the second case, the ball will rise as the water is added.

Discussion of the task: Now we should discuss how the resulting solution depends on the values of the parameters ρ, ρ_u and ρ_l. Pupils should realize that the outcome of the solution does not depend on the volume V of the ball. They should also think about the conditions under which the task makes sense from a physical point of view ($\rho_u < \rho < \rho_l$). The teacher can invite pupils to set parameters values for other substances and find the solution for them. For example, for given density ρ_l of the lower fluid, the result of the solution depends just on the density ρ_u of the upper fluid. Indeed, if we denote by $\overline{\rho_u}$ the density of a new added upper fluid, the ball will rise if

$$\frac{\rho_l - \rho}{\rho_l - \rho_u} < \frac{\rho_l - \rho}{\rho_l - \overline{\rho_u}}$$

From here we obtain the condition $\overline{\rho_u} > \rho_u$, i.e. the ball will rise in any case if the density of the new added upper fluid is greater than the density of the original one. One can see that the result does not depend on the density of the ball. The situation when the parameter ρ_l is fixed can be discussed using the similar reasoning. Pupils should come to the conclusion that the ball will rise in any case if the density of the new lower fluid is greater than the density of the original one regardless of the density of the ball. The discussion of the more complicated case when the both parameters ρ_u and ρ_l are changed can be done as well.

Conclusion

The examples above show how it can be advantageous to consider physical formulas as parameter relationships. Appropriate parameters usage is a powerful tool to solve physical problems that can be addressed effectively. The parameterization of the physical tasks also hides a great didactic potential. It allows a discussion between teacher and pupils on individual cases of problem solving depending on parameter values. Thus, pupils can better understand the physical formulas and learn to use them actively. The teacher should not be content with the fact that pupils only "solve" the example. He should also lead them to think about the relationships between the physical quantities, about the individual steps of the solution, and to try to understand them. This activity is very enriching for the pupils.

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