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# A CONCEPTUAL FRAMEWORK FOR EXAMINING MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN THE CONTEXT OF SUPPORTING MATHEMATICAL THINKING

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**Abstract:**
This study has been aimed to propose a conceptual framework that helps researchers examine mathematics teachers' PCK in the context of supporting students' mathematical thinking. "*Advancing Children's Thinking Framework*" which is a pedagogical model developed by Fraivillig, Murphy and Fuson (1999) that supports the development of students' conceptual understanding of mathematics has been adopted as the theoretical foundation. Pedagogical content knowledge (knowledge of students' thinking and knowledge of instructional strategies and representations) has been examined in the context of supporting mathematical thinking and has been interconnected to Advancing Children's Thinking Framework. Then, a new framework has been obtained. Instructional examples included within the framework suggested as a result of the interconnection have become the indicators regarding PCK of mathematics teachers in the context of supporting mathematical thinking. Some examples from a performed research where this framework has been used as an analytical framework have been presented. As a conclusion, it can be said that the suggested framework may be a useful tool for the researchers and teacher educators who are dealing with teachers' knowledge focusing on students' mathematical thinking and a guide for the teachers.

**Keywords:** conceptual framework, pedagogical content knowledge, mathematics teachers, mathematical thinking

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## Introduction

Effective teaching requires teachers to have specific knowledge and skills. The starting point of many research aimed at determining the knowledge that a teacher needs to have is Shulman's studies. Shulman (1986) suggested that a person who knows something does not mean that this person can teach this issue. According to Gearhart & Saxe (2004), teachers who have knowledge about the subject and have flexible pedagogical knowledge are called perfect teachers. This calls the concept of pedagogical content knowledge (PCK) to our mind.

Shulman (1987) has defined PCK as a special amalgam of subject matter knowledge and pedagogical knowledge. PCK is described as the most beneficial representations, the most powerful metaphors/analogies as well as best examples and explanations used to make a subject of a special field understandable to others (Shulman, 1987). Subsequent to Shulman's definition, PCK has been discussed and examined by many researchers (Grossman, 1990; Fennema & Franke, 1992; Magnusson, Krajcik & Borko, 1999; An, Kulm & Wu, 2004; Ball, Thames & Phelps, 2008). The agreed components for PCK that has been modeled through different components by several researchers can be listed as knowledge of students' thinking, knowledge of instructional strategies, knowledge of curriculum, content knowledge and knowledge of assessment. It is known that many studies have been carried out where the PCK of mathematics teachers has been analyzed within the context of one or a more of these components.

An, Kulm & Wu (2004) emphasized that deep and broad PCK is important and necessary for effective teaching. We can say that many researchers such as An, Kulm & Wu have agreed upon the fact that a more effective and quality teaching depends on teachers' PCK. In other words, it is possible to assume that teachers with improved PCK could be more successful in achieving the goals of teaching mathematics and providing a meaningful mathematics education for students. So, teachers' PCK can also be considered as an unavoidable component when providing students with mathematical thinking skill and supporting this way of thinking which is one of the objectives of mathematics teaching in our country (Ministry of National Education [MNE], 2011). In fact, it won't be easy to develop students' mathematical thinking if a teacher is unable to understand how his/her students comprehend a particular issue and is unable to estimate what type of misconceptions he/she will have, or which strategies he/she has to refer to in particular cases. In National Council of Teachers of Mathematics (NCTM) (2000) Standards, it has been emphasized that effective teaching involves observing the students, paying close attention to the thoughts and explanations of students, having mathematical objectives and using knowledge when taking instructional decisions.

Teachers using these practices motivate their students to mathematical thinking and reasoning and provide learning opportunities for students at every level of understanding that will challenge them (NCTM, 2000: 19). Therefore, the PCK of a teacher is one of the concepts that have to be considered as first priority when it comes to supporting/developing students' mathematical thinking.

Teachers who are knowledgeable of the behaviour of their students have more flexibility, capacity and creativity in constructing lessons and tasks that meet student learning needs (Lee, 2006: p. 1-2). Professional development programs that focus on students' mathematical thinking have produced results that consistently indicate the value of the approach for both students and teachers (Norton, McCloskey & Hudson, 2011). Research projects such as Cognitively Guided Instruction (ter et al., 1989), the Purdue Problem-Centered Mathematics Project (Cobb, Wood and Yackel, 1990; Cobb et al. 1991), SummerMath (Simon and Schifter, 1991), the Kenilworth Project (Maher, Davis and Alston, 1991, 1992; Maher and Martino, 1992), the Mathematics Case Methods Project (Barnett, 1998), the work of Gordon and MacInnis (1993) and the work of Putnam and Reineke (1993) have found the following to be of potential benefit for both teachers and students when teachers tend to their students' mathematical thinking:

- The ability on the part of teacher to construct or select appropriate, worthwhile mathematical tasks;
- A shift from teacher-centered didactical instruction to student-centered problem-solving instruction;
- Higher levels of conceptual understandings by students without compromising their computational performances;
- More positive beliefs of teachers and students toward mathematics (cited in Chamberlin, 2002: p. 1-2)

Although there are numerous research that examine the pre/in-service teachers' knowledge based on the knowledge of students or researches that examine the knowledge of students' thinking in relation to mathematical thinking (An, Kulm & Wu, 2004; Jenkins, 2010; Kılıç, 2010, 2011; Lee, 2006; Norton, McCloskey & Hudson, 2011; Sleep & Boerst, 2012; Yeşildere-İmre & Akkoç, 2012), studies that examine a model regarding mathematical thinking within the scope of PCK are limited. An, Kulm & Wu (2004) classified the knowledge of students' thinking into four categories in their studies which they aimed to compare to the PCK of the middle school mathematics teachers in America and China: Addressing students' misconceptions, engaging students into math learning, promoting students' thoughts regarding mathematics, and building on students' math ideas. On the other hand, Lee (2006) built a conceptual framework in order to analyze teachers' knowledge of middle school students' mathematical thinking of algebraic word problem solving therefore benefiting from the study of An, Kulm & Wu (2004). In another study, Cengiz, Kline & Grant (2011) built a framework (Extending Student Thinking Framework) by gathering Advancing Children's Thinking model built by Fraivillig, Murphy and Fuson (1999) and other studies together. They examined this framework within the scope of Mathematical Knowledge for teaching developed by Ball, Thames and Phelps (2008) and focused on whole-group discussions based on students' existing mathematical thinking. As it is seen, by examining PCK and mathematical thinking all together it can help with the search for an answer to the question "*What type of knowledge should a teacher have who wants to support/develop students' mathematical thinking and what should he/she do for this?".* Although current studies help to answer this question, there is still a need for deeper studies regarding this issue. Based upon this idea, mathematics teachers' PCK has been examined within the context of supporting mathematical thinking in this study. Therefore, the purpose of this study is to propose a conceptual framework that helps researchers examine mathematics teachers' PCK in the context of supporting students' mathematical thinking.

In this study, examination of mathematics teachers' PCK is attempted within the context of supporting mathematical thinking. Supporting/developing students' mathematical thinking has been considered as the most significant idea that forms the theoretical foundation of the study. "*Advancing Children's Thinking Framework*" which is a pedagogical model developed by Fraivillig, Murphy and Fuson (1999) that supports the development of conceptual understanding of mathematics by students has been adopted as the theoretical base.

This model has been preferred, because it does not only suggest that students' mathematical thinking should be supported and developed, but also shows a concrete way as to how teachers can do this. Another theoretical idea that has been adopted as a base in the study is Shulman's, (1986, 1987) the idea of PCK. The focal points of this study are the knowledge of students' thinking as well as knowledge of instructional strategies and representations components of PCK. In the next part, first of all, each adopted theoretical framework will be introduced and explanations regarding the conceptual framework that has been built by associating to these will be presented. Afterwards, examples from performed research where this framework has been used as an analytical framework will be presented.

**Mathematical Thinking**

One of the skills that are aimed to make the students gain in mathematics teaching is mathematical thinking (MNE, 2011). Stacey (2008) has specified the importance of mathematical thinking in three ways:

1) Mathematical thinking is an important goal of schooling,
2) Mathematical thinking is important as a way to learn mathematics,
3) Mathematical thinking is important for teaching mathematics.

When the learning and mathematical thinking are examined together, it can be said that many components come to the forefront. Schoenfeld (1992: 5) has listed the fundamental aspects of mathematical thinking as core knowledge, problem-solving strategies, and effective use of resources, having a mathematical perspective and engaging in mathematical practices. Mathematics teaching should present practices that develop a student's knowledge in each of these fields (Swan & Ridgway, 2002).

Many people think/may think that mathematical thinking is a way of only thinking related to mathematics. According to Burton (1984), this type of thinking is mathematical not only because it is about mathematics, but because the operations it is based on are mathematical operations and its field of application is general. Therefore, regardless of a person being a mathematician or not, all individuals use mathematical thinking in their lives, in the events or facts they are confronted with or in solving problems. In other words, mathematical thinking is not a way of thinking peculiar to only mathematicians. On the contrary, it's a way of thinking that each person having a profession should use it at the present time (Alkan & Bukova-Güzel, 2005). Consequently, individuals use mathematical thinking in every phase of their lives or to solve their problems wittingly or unwittingly (Arslan & Yıldız, 2010).

On the other hand, NCTM (2000) points out the increase in the mathematical level that is necessary for individuals in the workplace, in professional areas ranging from health care to graphic design and also the increase in mathematical thinking and problem-solving levels. According to Umay (2003), one of the fields (possibly the first one) where thinking skills and logic is used intensively is mathematics. One of the objectives of learning mathematics should not only be learning mathematical terms, concepts and language of mathematics, but learning to think by using them (Umay, 2007). To put it more clearly, practicing mathematics is a way of thinking beyond using ample formulas, keeping technical data in mind and re-proofing an already proven theorem (Yıldırım, 2004). For this reason, the desired mathematical education is the one that prioritizes the students to gain thinking, reasoning, problem-solving skills and the ability to relate these to daily life while obtaining mathematical knowledge (Umay, 2003).

Mathematical thinking is also emphasized in the standards and programs developed for the learning and teaching processes of mathematics. One of the standards suggested by NCTM (1991: 21) regarding mathematics teaching is 'To provide students with mathematical thinking skills'. This objective has also been included within the general objectives of mathematics education within the scope of the renewed mathematics lesson curriculum in our country and mathematical thinking skills have been determined as one of the skills that the curriculum aims to develop (MNE, 2011). The expression of *"Besides gaining basic concepts and skills, learning mathematics also involves thinking mathematically, developing general problem-solving strategies, maintaining a positive attitude towards mathematics and understanding that mathematics is an important tool used in real life"* is one part of the MNE (2011) mathematics lesson curriculum that brings emphasis to the curriculum places on mathematical thinking. In addition, it has been indicated that the activities brought to the class by the teachers (within the scope of mathematics lesson curriculum) should be aimed at providing the students with high-level mathematical thinking skills such as analysing, synthesising, assessment, connection, classification, generalization and deduction (MNE, 2005).

NCTM objectives have shown a change from its traditional practice that was summarizing the required mathematical outputs such as skills, concepts and practices knowledge through wider trends, attitudes and beliefs regarding the nature of mathematical knowledge and own mathematical thinking of the individual (Romberg, 1994). Expectations and objectives aimed at developing the mathematical thinking skills of students arise accordingly. These expectations and objectives can only be put into practice within teaching environments composed of teachers carrying out effective teaching. For this reason, it is in evidence that teachers play a significant role in supporting/developing students' mathematical thinking.

**Advancing Children's Thinking Framework**

Fraivillig, Murphy and Fuson (1999) have emphasized that teachers should consider the components of eliciting students' solutions, supporting students' conceptual understanding and extending their mathematical thinking in an instruction where students' mathematical thinking are supported and developed. In this direction, they have developed the "*Advancing Children's Thinking Framework*" which is a pedagogical model that supports the development of students' conceptual understanding of mathematics. This model is composed of three components: eliciting students' solutions, supporting their conceptual understanding and extending their mathematical thinking.

The instruction of eighteen mathematics teachers has been observed in the study. Lessons of six teachers (that are characterized as qualified) have been monitored through extra observations and one teacher has been examined as special case. Instructional strategies that the teachers refer in advancing students' mathematical thinking have been listed within the scope of data. Things that a teacher can do in order to develop students' mathematical thinking in a questioning class environment where the thoughts and solutions of students are found are presented in Table 1.

| **Eliciting** | **Supporting** | **Extending** |
|---|---|---|
| Facilitates students' responding | Supports describers' thinking | Maintains high standards and expectations for all students |
| Elicits many solution methods for one problem from the entire class | Reminds students of conceptually similar problem situations | Asks all students to attempt to solve difficult problems and to try various solution methods |
| Wait for and listen to students' descriptions of solution methods | Provides background knowledge | Encourages mathematical reflection |
| Encourages elaboration of students' responses | Directs group help for an individual student | Encourages students to analyse, compare, and generalize mathematical concepts |
| Conveys accepting attitude toward students' errors and problem solving efforts | Assists individual students in clarifying their own solution methods. | Encourages students to consider and discuss interrelationships among concepts |
| Promotes collaborative problem solving | Supports listeners' thinking | Lists all solution methods on the chalkboard to promote reflection |
| Orchestrates classroom discussions | Provides teacher-led instant replays. | Goes beyond initial solution methods |
| Uses students' explanation for lesson's content | Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method | Pushes individual students to try alternative solution methods for one problem situation |
| Monitors students' levels of engagement | Supports describers' and listeners' thinking | Promotes use of more efficient solution methods for all students |
| Decides which students need opportunities to speak publicly or which methods should be discussed | Records symbolic representation of each solution method on the chalkboard | Uses students' responses, questions, and problems as core lesson |
| | Asks a different student to explain a peer's method | Cultivates love of challenge |
| | Supports individuals in private help sessions | |
| | Encourages the students to request assistance (Only when needed) | |

**Table 1:** Examples of Instructional Strategies of ACT Framework
(Adapted from Fraivillig, Murphy & Fuson, 1999, p. 155)

As it is seen, the model aimed at advancing students' mathematical thinking is composed of three components as eliciting, supporting and extending. "Eliciting" is considered to enable the students to explain their thoughts. Knowing what students think and finding out their answers is considered significant in supporting students' thinking. Yackel (1995) argued the reason for this as the teacher can provide learning opportunities for all the students by these means (as cited in Fraivillig, Murphy & Fuson, 1999, p. 149).

The supporting component of the model involves encouraging the students to explain their own solutions that they bring out through their current cognitive abilities and the teacher to take pedagogical decisions in this direction. Instructional components of eliciting and supporting involve instructional strategies aimed at students to reach their thoughts regarding solutions that they are familiar with and easing this process. However, these components do not involve the methods that teachers refer to in order to challenge and extend students' thinking. Extending, which is the last component addresses the strategies that could be used to advance the students' progress through their zones of proximal development (Fraivillig, Murphy & Fuson, 1999)

**Pedagogical Content Knowledge (PCK)**

Shulman (1987) has defined PCK as knowledge of teaching where subject matter knowledge intersects with pedagogical knowledge and where practice knowledge integrates with theoretical knowledge. According to Shulman (1987), PCK, is the knowledge that differs a specialist in a particular field (for instance, a mathematician) from an educationist (mathematics teacher). Fennema and Franke (1992) have emphasized the important aspects of PCK in their definitions for teacher knowledge:

> *"Knowledge of mathematics teaching includes knowledge of pedagogy, as well as understanding the underlying processes of the mathematical concepts, knowing the relationship between different aspects of mathematical knowledge, being able to interpret that knowledge for teaching, knowing and understanding students' thinking, and being able to assess student knowledge to make instructional decisions."* (p. 161)

Shulman (1986) defined PCK as the most useful forms of representation of a subject, the most powerful analogies, illustrations, examples, explanations and demonstrations. In other words, the knowledge is used to represent the subject to make it comprehensible to others. In addition, he included what makes it easy or difficult to learn of specific concepts, especially knowledge regarding the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning within the scope of PCK. Based on Shulman's definition, Kovarik (2008) has divided PCK into two categories and sub-categories as ways of knowledge of representations and approaches and knowledge of student thinking. When the models examining PCK are analyzed, it is seen that these two components which are prominent in Shulman's definition and which Kovarik has emphasized are examined within the scope of PCK with different names by many different researchers. In this study, it has been decided to examine PCK within the scope of knowledge of students' thinking and knowledge of instructional strategies and representation components.

Knowledge of students' thinking involves knowing what makes it easy or difficult to learn specific concepts (Ball et al., 2008; Shulman, 1986), to know how students perceive a concept and how they think (Ball et al., 2008; Fennema & Franke, 1992), to determine the misconceptions and learning disabilities of students (An, Kulm & Wu, 2004; Fennema & Franke, 1992; Kovarik, 2008; Magnusson, Krajcik & Borko, 1999; Park & Oliver, 2008; Schoenfeld, 1998; Shulman, 1986), and to be aware of the prior knowledge of students (Kovarik, 2008; Magnusson et al., 1999; Schoenfeld, 1998; Shulman, 1986). Knowledge of instructional strategies and representation involves the demonstrations, activities and examples that the teacher will use and the strategies peculiar to the topic and subject (Ball et al., 2008; Kovarik, 2008; Magnusson et al., 1999; Park & Oliver, 2008; Shulman, 1986). How the knowledge of students' thinking has been defined and under which names they have been examined within the frame of the analyzed models have been summarized in Table 2.

| Researcher | Component | Content of the Component |
|---|---|---|
| **Shulman (1987)** | Knowledge of Learners and Their Characteristics | An understanding of what makes the learning of spesific topics easy or difficult |
| | | The conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons |
| | | Preconceptions and misconceptions (Shulman, 1986). |
| **Grossman (1990)** | Knowledge of Students' Understanding | |
| **Fennema & Franke (1992)** | Knowledge of Learners Cognitions in Mathematics | Knowledge of how students think and learn |
| | | Understanding the processes the students will use and the difficulties and successes that are likely to occur |
| | | Knowledge of how students acquire the mathematics content |
| **Magnusson, Krajcik & Borko (1999)** | Knowledge of Students' Understanding of Science | Knowledge of requirements for learning (prequisiste knowledge, abilities and skills tahta students might need, ability levels or different learning styles) |
| | | Areas of student difficulty |
| **An, Kulm &Wu (2004)** | Knowing Students' Thinking | Building on students' mathematical ideas |
| | | Addressing students' misconceptions |
| | | Engaging students' in mathematics learning |
| | | Promoting students' thinking mathematically |
| **Ball, Thames & Phelps (2008)** | Knowledge of Content and Students | Anticipating what students are likely to think and what they will find confusing |

| | | |
|---|---|---|
| | | Predicting what students will find interesting and motivating when choosing an example |
| | | Anticipating what students are likely to do with it and whether they will find it easy or hard when assigning a task |
| | | Knowledge of common student conceptions and misconceptions about particular mathematical content |
| **Park & Oliver (2008)** | Knowledge of Students' Understanding | Misconceptions |
| | | Learning difficulties |
| | | Motivation and interest |
| | | Need |
| **Kovarik (2008)** | Knowledge of Student Thinking | Student Prior Knowledge |
| | | Mathematical Background |
| | | Student Misconceptions |
| | | Conneceting Prior Knowledge to New Knowledge |
| | | Anticipating Students Questions |
| | | Assessing Understanding |

**Table 2**: Knowledge of Students' Thinking in Different PCK Frameworks

It is seen that knowledge of students' thinking takes place in teacher knowledge models with different terms such as knowledge of learners and their characteristics, knowledge of students' understanding, knowledge of learners' cognitions in mathematics, knowing students' thinking and knowledge of student thinking. When the definitions are examined, although there are some varying points, it is predominantly seen that a similar scope is pointed out. The sub-components of knowledge of students' thinking could be listed as follows based on the relevant literature: determining students' current knowledge, connecting prior knowledge to new knowledge, knowing students' misconceptions, valuing students' questions and thoughts, foreseeing students' thoughts and considering students' individual differences.

**Determining students' current knowledge**: One of the components required for the teacher to support students' mathematical thinking is to firstly determine the students' current knowledge. Knowing the current situation of the students helps the teacher to take instructional decisions and to plan his/her instruction (Fennema & Franke, 1992). Shulman (1986) has indicated that students at different ages with different knowledge bring some previous knowledge with them and that these should be known by the teacher as there is a high possibility of this prior knowledge transforming into misconception later. For this reason, determining previous knowledge and precognitions of students by the teacher is considered extremely important.

**Connecting prior knowledge to new knowledge:** According to Bransford et al. (2000) students use their prior knowledge to understand and configure the new ones and sometimes this knowledge may cause the new knowledge to be misinterpreted (as cited in Kovarik, 2008: p. 32). In other words, students build new knowledge upon the previous knowledge. For this reason, it is quite important that the prior knowledge of students is configured correctly. It would be beneficial for the students having teachers who relate the prior knowledge of students with new knowledge, so that the students can configure their mathematical thinking correctly.

**Knowing students' misconceptions:** Determining the misconceptions of students, knowing the source of misconceptions and referring to ways that remove these is another most important component which teachers should consider in supporting students' mathematical thinking. Examining misconceptions has caught the attention of many researchers and they have shown great effort in finding the sources of misconceptions (Even & Tirosh, 2008). In their research, An, Kulm and Wu (2004) have found that teachers are using various activities, graphics, manipulatives and processes in order to correct misconceptions and are focusing on use of concrete models for configuring abstract thoughts. Knowing the misconceptions of students is a necessary component for teachers in supporting the students to configure their mathematical thinking correctly.

**Valuing students' questions and thoughts:** Considering the questions of students is also an important component that will direct teachers' instructions. Park and Oliver (2008) suggested that the questions asked by the students are one of the factors which affect the development of teachers' PCK. According to the researchers, challenging questions asked by students deepen and expand the subject matter knowledge of the teacher. According to NCTM (2000), paying close attention to the thoughts and explanations of students is one of the necessities for effective teaching. The teacher should listen to students' answers and should try to understand the students' thinking when he/she asks a question to the students or wants an explanation from the students. This component is also included within the Fravillig, Murphy and Fuson's model as one of the strategies that a teacher can use to elicit the students' thoughts.

**Foreseeing students' thoughts:** According to Ball et al. (2008), teachers should predict what students are thinking and what they see as confusing. They should also foresee what would be interesting and motivating for students when they choose an example. They should predict using tools where students can participate when they carry out an activity and they should predict if that activity would be easy or difficult for the students (Ball et al., 2008). It can be said that a teacher who acts by foreseeing the thoughts of students can make the students the focal point and plan his/her instruction in this manner. Thus, this would be an important step in supporting students' mathematical thinking.

**Considering students' individual differences:** Park and Oliver (2008) have emphasized the importance of the abilities, learning styles, development levels and different needs of students within the scope of knowledge regarding the learning of students in their PCK model. Magnusson et al. (1999) have indicated the required abilities, skills, students' needs and learning styles as the requirements of learning. This component has been named as 'considering students' individual differences' in the study presented.

Knowledge of instructional strategies and representations which have been emphasized by Shulman as another component of PCK has been defined as one of the knowledge components which a teacher should have by many researchers. Table 3 shows under which name and scope this component has been examined in the literature.

| Researcher | Component | Content of the Component |
|---|---|---|
| **Shulman (1986)** | | Representations |
| | | The most powerful analogies, illustrations, examples, explanations and demonstrations |
| **Grossman (1990)** | Knowledge of Instructional Strategies | |
| **Magnusson, Krajcik & Borko (1999)** | Knowledge of Instructional Strategies | Subject-specific strategies |
| | | Topic-specific strategies (Representations, activities) |
| **Ball, Thames & Phelps (2008)** | Knowledge of Content and Teaching | Choosing which examples to start with and which examples to use to take students deeper into the content |
| | | Knowing the instructional advantages and disadvantages of representations used to teach a specific idea |
| | | Identifying what different methods and procedures afford instructionally |
| **Park & Oliver (2008)** | Knowledge of Instructional Strategies | Subject-specific strategies |
| | | Topic-specific strategies |
| **Kovarik (2008)** | Knowledge of Representaions and Approaches | Demonstrations (Graphs, Tables, Formulas) |
| | | Examples (Real World Examples, Problems) |
| | | Analogies |

**Table 3:** Knowledge of Instructional Strategies and Representations in Different PCK Frameworks

While some researchers have examined representations within the scope of the component named 'instructional strategies', some of them have given place to the representations by the name of the said component. Although the component names defined by the researchers differentiate, it can be said that they are considerably similar. Magnusson et al. (1999) have examined knowledge of instructional strategies in two categories; as subject-specific strategies and topic-specific strategies. Subject-specific strategies represent the general approaches used to teach a particular area (science, mathematics etc.). Teachers' knowledge of subject-specific strategies involves the ability to define and realize a strategy and its phases. According to Magnusson et al. (1999), topic-specific strategies are strategies that may be used to help the students understand certain (science) concepts. This strategy has been divided into two; as representations and activities. Knowledge of representations is the knowledge regarding the ways of representing certain concepts or principles used to ease the learning process of students. Examples of representations are illustrations, examples, models and analogies. Activities peculiar to the subject are activities that may be used to make the students comprehend certain concepts or relationships. Examples of such activities are problems, demonstrations, simulations, researches and experiments (Magnusson, Krajcik & Borko, 1999).

Kovarik (2008) has indicated knowledge of representations and approaches as a component of PCK in the model she has developed based on Shulman's PCK definition. Kovarik has divided knowledge of representations and approaches as demonstrations, examples and analogies. Demonstrations include graphics, tables and formulas. Examples are real-world examples and problems. Analogies have also been included within the knowledge of representations and approaches.

As it is seen, these researchers have also examined knowledge of instructional strategies and representations in a similar way in terms of scope. Kovarik's knowledge of representations and approaches classification corresponds to topic-specific strategies definition of Magnusson et al. (1999).

As a result, PCK has been examined within the scope of knowledge of students' thinking and knowledge of instructional strategies and representations components in this study. Six sub-components (that have been summarized as a result of literature review) have been defined for knowledge students' thinking. The classification suggested by Kovarik (2008) has been used for knowledge of instructional strategies and representations. In this way, the PCK framework adopted in the study and presented in Table 4 has been obtained.

| **PEDAGOGICAL CONTENT KNOWLEDGE** | |
|---|---|
| **Knowledge of Students' Thinking** | Knowledge of Instructional Strategies and Representations |
| **Determining students' current knowledge** | Representations |
| **Connecting prior knowledge to new knowledge** | Examples (Real life examples-Problems) |
| **Knowing students' misconceptions** | Analogies |
| **Valuing students' questions and thoughts** | |
| **Foreseeing students' thoughts** | |
| **Considering students' individual differences** | |

**Table 4:** PCK Framework Adopted in this Study

The theoretical framework chosen for PCK in the study is presented in Table 4. But it is known that teacher knowledge is not monolithic, it is a large, integrated, functioning system with each part difficult to isolate (Fennema & Franke, 1992). Park and Oliver (2008) indicated that an improvement in one of the components of PCK will affect the other components and that only in this way an improvement can be provided in the whole PCK. In addition to this, they have suggested that PCK can be seen as a combination of other components of teacher knowledge. It is considered that improvement in only one component of PCK may not provide much benefit to the teacher and lack of compliance between the components would cause trouble in terms of PCK development of the teacher (Harel & Lim, 2004; Park & Oliver, 2008). For this reason, the interaction between the components of PCK should always be taken into consideration and other fields of PCK should also be considered when examining knowledge of students' thinking, knowledge of instructional strategies and representations components. However, the theoretical aspect of this study has been limited by the two components of PCK mentioned. The reason for this is the idea that the teacher prioritizes these two types of knowledge during instruction that supports mathematical thinking and also the difficulty of studying on all the components of PCK.

## Our Conceptual Framework: PCK in the context of Supporting Students' Mathematical Thinking Framework

Finding a concrete answer to the question "*What type of knowledge should a teacher have who wants to support/develop students' mathematical thinking and what should he/she do for this?*" is hard for mathematics teachers and mathematics educators. The main purpose of this study is to make a contribution to the field in terms of finding an answer to this question. For this purpose, mathematics teachers' PCK has been examined theoretically in the context of supporting students' mathematical thinking and the theoretical frameworks chosen for PCK has been interconnected to mathematical thinking. The interconnected framework has been started to derive from ACT Framework (1999) of Fraivillig, Murphy & Fuson composed of three components including examples of instructional practices. Then the PCK model modified reflecting onto Shulman's model has been integrated on this model. Each instructional example included within the ACT has been re-examined within the context of two components of PCK (knowledge of students' thinking and knowledge of instructional strategies and representations) and the sub-components of these components. While making this interconnection, first of all PCK components were placed horizontally and three components of ACT were placed vertically on a table. Afterwards, instructional practices within ACT were placed to the appropriate cell in this table (considering within the scope of sub-components of PCK). While the same component has been placed in more than one cell sometimes, no component has been assigned for some cells. In this way, the theoretical frameworks shown in Table 5 and Table 6 have been obtained. For instance, determining students' current knowledge component of PCK has been interconnected to eliciting and supporting steps of ACT. No relationship has been established for extending step. According to the interconnection made; a teacher who wants to determine students' current knowledge should ask the students to explain their own solutions and listen to them, encourage the students to explain their answers in detail, decide which students should be provided with answering opportunities in front of the class and ask these students to explain their thoughts, listen to them and share the students' thought/solution with the whole class within the scope of "*Eliciting*". In terms of "*Supporting*", it is considered that a teacher who is trying to determine students' current knowledge can assist them when explaining their solutions or thoughts on an individual basis.

Since interconnection has been started with ACT, one component of ACT might be placed under more than one component of PCK. For instance, asking students to explain their solutions and listen to them which are some of the eliciting components of ACT have been examined within the scope of both determining students' current knowledge and knowing the misconceptions component of PCK.

| PCK Component / Components of ACT | Knowledge of Students' Thinking | | | | | |
|---|---|---|---|---|---|---|
| | Determining students' current knowledge | Connecting prior knowledge to new knowledge | Knowing students' misconceptions | Valuing students' questions and thoughts | Foreseeing students' thoughts | Considering students' individual differences |
| ELICITING | Wait for and listen to students' descriptions of their solution methods<br><br>Encourages students to explain their responses in detail<br><br>Decides which students need opportunities to speak in front of the class and requesting these students to explain their thoughts<br><br>Shares a student's thought/solution with all class | | Wait for and listen to students' descriptions of their solution methods<br><br>Encourages students to explain their responses in detail<br><br>Uses students' misconceptions (that are determined through their explanations) for lesson's content | Uses students' thoughts/questions for lesson's content<br><br>Conveys accepting attitude toward students' errors and problem solving efforts | Directing lesson's content by predicting what students will find easy or confusing through their explanations | Elicits many solution methods for one problem from the entire class<br><br>Conveys accepting attitude toward students' errors and problem solving efforts<br><br>Decides which students need opportunities to speak in front of the class |
| SUPPORTING | Assists individual students in clarifying their own thoughts or solution methods | Reminds students of conceptually similar problems/ situations<br><br>Provides background knowledge<br><br>Provides teacher-led instant replays | Reminds students of conceptually similar problems/ situations<br><br>Assists individual students in clarifying their own thoughts or solution methods | Asks a different student to explain a peer's solution method | | Encourages the students to request assistance (when needed) |
| EXTENDING | | Encourages students to consider and discuss interrelationships among concepts | Encourages students to analyse, compare, and generalize mathematical concepts in terms of removing the misconceptions | Uses students' creative and different responses, questions, and problems as core lesson | | |

**Table 5: Conceptual Framework of The Research (Connecting ACT Components to Knowledge of Students' Thinking)**

| **PCK Components** / **Components of ACT** | **Knowledge of Instructional Strategies and Representations** | | | |
|---|---|---|---|---|
| | Representations | Examples | | Analogies |
| | | Real Life Examples | Problems | |
| **ELICITING** | Wait for and listen to students' detailed descriptions of the representations in their solution methods<br><br>Encourages students to explain the representations that they used in their solution methods in detail<br><br>Shares students' representations that they used in their solution methods with all class<br><br>Uses students' representations and explanations for lesson's content | Wait for and listen to students' real life examples<br><br>Shares students' real life examples with all class<br><br>Encourages students to explain their real life examples in detail<br><br>Uses students' real life examples for lesson's content | Elicits many solution methods for one problem from the entire class<br><br>Wait for and listen to students' descriptions of their solution methods<br><br>Shares students' solutions with all class<br><br>Encourages students to explain their responses in detail<br><br>Promotes collaborative problem solving | Wait for and listen to students' analogies<br><br>Encourages students to explain their analogies in detail<br><br>Shares students' analogies with all class |
| **SUPPORTING** | Reminds students of similar representations<br><br>Provides prior representations<br><br>Provides teacher-led instant replays about using representations<br><br>Assists individual students in clarifying their own representations<br><br>Demonstrates teacher-selected representations without endorsing the adoption of a particular representation)<br><br>Asks a different student to explain a peer's representation in her/his solution method | Reminds students of conceptually similar real life examples in problems/ situations<br><br>Provides prior real life examples<br><br>Provides teacher-led instant replays about real life examples<br><br>Assists individual students in clarifying their own real life examples<br><br>Demonstrates teacher-selected real life examples without endorsing the adoption of a particular example)<br><br>Asks a different student to explain a peer's real life example | Reminds students of conceptually similar problems/ situations<br><br>Assists individual students in clarifying their own thoughts or solution methods<br><br>Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method<br><br>Asks a different student to explain a peer's method<br><br>Demonstrates an alternative solution method for one problem | Reminds students of conceptually similar analogies<br><br>Provides prior analogies<br><br>Provides teacher-led instant replays about analogies<br><br>Assists individual students in clarifying their own analogies<br><br>Demonstrates teacher-selected analogies without endorsing the adoption of a particular analogie)<br><br>Asks a different student to explain a peer's analogie |
| **EXTENDING** | Asks all students to attempt to solve difficult problems and to try using various representations<br><br>Promotes use of more efficient representations in the solution methods for all students<br><br>Encourages students using representations to analyze, compare, and generalize mathematical concepts<br><br>Encourages students using different representations to consider and discuss interrelationships among concepts<br><br>Lists all representation in students' solution methods on the chalkboard to promote reflection<br><br>Uses students' creative and different responses, questions, and problems as core lesson | Encourages students using real life examples to analyse, compare, and generalize mathematical concepts<br><br>Promotes use of more efficient real life examples in the solution methods for all students<br><br>Lists all real life examples in students' solution methods on the chalkboard to promote reflection<br><br>Uses students' creative and different real life examples as core lesson | Asks all students to attempt to solve difficult problems and to try various solution methods<br><br>Promotes use of more efficient solution methods for all students<br><br>Encourages students to analyze, compare, and generalize mathematical concepts for encountered problem<br><br>Encourages students to consider and discuss interrelationships among concepts<br><br>Lists all solution methods on the chalkboard to promote reflection<br><br>Uses students' responses, questions, and problems as core lesson<br><br>Cultivates love of challenge | Encourages students using analogies to analyze, compare, and generalize mathematical concepts<br><br>Lists all analogies on the chalkboard to promote reflection<br><br>Encourages students using different analogies to consider and discuss interrelationships among concepts |

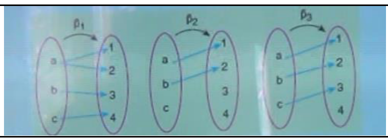
**Table 6: Conceptual Framework of The Research (Connecting ACT Components to Knowledge of Instructional Strategies and Representations**

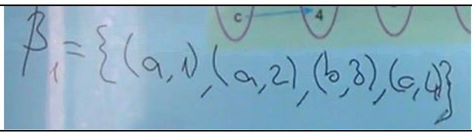
**Using "PCK in the context of Supporting Students' Mathematical Thinking Framework"**

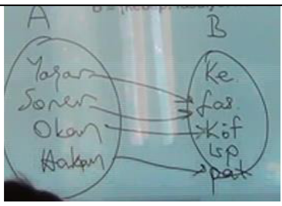
Examples from the results of a research (Tataroğlu Taşdan, 2014) where the above-mentioned interconnected conceptual framework has been used as an analytical framework will be given in this section. Mathematics teachers' PCK development (in the context of supporting mathematical thinking) has been examined in the research mentioned.

The study was a qualitative research carried out by Tataroğlu Taşdan (2014) as a PhD Thesis. The aim of the research was to improve mathematics teachers' PCK in the context of supporting students' mathematical thinking. In the research that has been carried out with six volunteer teachers, mathematics teachers' teaching of function concept have been observed for two years consecutively prior to and after the implementation (a workshop, meetings, interviews). Observations have been recorded on video. All the video recordings were watched before starting the analyses. The actions done by the teacher in the in-class practices have been taken into consideration in order to decide which category within the framework these will be included. At this stage, some of the components have been divided into two and some of them have been renamed when considered necessary. For instance, "*reminding prior knowledge*" component differs depending on who has reminded it and since this is important within the scope of the research, it has been deemed more suitable to examine this sub-component as two sub-components as "reminding prior knowledge by the teacher" and "*teacher asking the student to remember the prior knowledge"*. In the analysis, the teacher's approach has been included in the suitable sub-component. However, it has been determined that there are negative approaches concerning this sub-component. With the thought that indicating these cases is necessary for reflecting the PCK of teachers, it has been decided to arrange these findings by classifying them as positive and negative. Negative findings show that a negative approach has been observed towards the teacher regarding that component or the teacher cannot use the opportunity positively although there is a very convenient classroom environment to establish a positive approach.

Transcribed lesson sections and some examples regarding the analysis of these scripts by the help of framework are shown below. Source of expression (teacher, student, blackboard, smartboard), expression, basic components of the framework, sub-components of the framework and some descriptions/notes have been included in the tables. The Stud. Abbreviation is used for student in the tables. The situations where students talk as a crowded group have been indicated as Stud. (together). In order to distinguish situations where the conversation belongs to the same student, indications such as Stud.1, Stud.2 has been used. Screen quotations have also been used where necessary. Explanations have been included (within the expressions) in square brackets and in italics in order to describe the current situation. Since some rows could not be included in any component within the scope of the framework, abbreviations as "…" have been used.

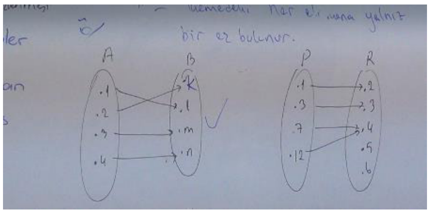
| Source of expression | Expression | Basic Components of the framework | Sub-components of the framework | Descriptions/ Notes |
|---|---|---|---|---|
| Ersin | ... Here are three relation diagrams. Let's look and try to see common and different properties of them. Ziya? | **ACT - Knowledge of Students' Thinking** Determining students' current knowledge - *Eliciting* | Wait for and listen to students' descriptions of their solution methods | *Teacher illustrated the Venn diagrams of three relations and asked the students to examine these relations.* |
| | | **ACT - Knowledge of Instructional Strategies and Representations** *Representations - Eliciting* | Demonstrates teacher-selected representations (without endorsing the adoption of a particular representation) | |
| Smart board |  | | | |
| Stud. | Teacher, the elements of the sets are the same. | | | |
| Ersin | The elements of the sets are the same. We can say this by looking at their common properties. | **ACT - Knowledge of Students' Thinking** Determining students' current knowledge- *Eliciting* | Shares a student's thought/solution with all class | *A student expressed his thought, then the teacher shared the student's thought with other students.* |
| Stud. | a goes to 1 in each one. | | | |
| Ersin | a goes to 1 in each one, okay. | **ACT - Knowledge of Students' Thinking** Determining students' current knowledge- | Shares a student's thought/solution with all class | *Teacher repeated the student's answer and shared with others.* |

| | | | | |
|---|---|---|---|---|
| | | *Eliciting* | | |
| Stud. | The elements of the relation diagrams are the same. | | | |
| Ersin | Are the elements of the relation diagrams the same? | **ACT - Knowledge of Students' Thinking** Determining students' current knowledge-*Eliciting* | Encourages students to explain their responses in detail | *The student's answer was wrong and the teacher provided whole students to think about this answer by asking a question.* |
| Stud. | No. | | | |
| Ersin | Let's write some of them. For example the elements of the relation $\beta_1$. | **ACT - Knowledge of Students' Thinking** Determining students' current knowledge-*Eliciting* | Wait for and listen to students' descriptions of their solution methods | *The teacher suggested to write the elements of the relation to help the students examine.* |
| Stud. | (a,1),(a,2),(b,3),(c,4)… | | | |
| Ersin | The elements of relation $\beta_1$ are [*the students are saying, the teacher is writing*] (a,1) ,(a,2), (b,3), (c,4). Okay. Now let's try to look them as a whole not only to the elements. | **ACT - Knowledge of Students' Thinking** Connecting prior knowledge to new knowledge-Extending | Encourages students to consider and discuss interrelationships among concepts | *Teacher encouraged the students to make generalization based on a spesific example.* |
| Smartboard |  | | | |
| Stud. | Teacher, if we calculate the number of subsets, all will be equal, $2^{12}$. | | | |
| Ersin | Yes, $2^{12}$ and we can say all are the subsets of the same set. What did we say when we defined the relation? We described relation as each subset of the Cartesian Product. Now we are focusing on some special ones. We will pass through to the function concept. Here are some similarities and differences. Try to see them. Try to consider the elements that are used or not used in the sets. Ata. | **ACT - Knowledge of Students' Thinking** Connecting prior knowledge to new knowledge-Supporting | Provides background knowledge | *When a student reminded the subset number, the teacher reminded previous knowledge and emphasized the focus and the purpose of the current discussion. So* |

| | | | | |
|---|---|---|---|---|
| | | | | *he prevented the discussion to go to an undesired way.* |
| Stud. (Ata) | Teacher, for instance some elements of the relations are common, for instance (a,1) is common for all but (c,4) is not. | | | |
| Ersin | Not common for all, okay. Let's look the elements of domain set. In relation $\beta_1$, Doğuş? | | | *The teacher interfered to direct the discussion.* |
| | ... | | | |
| Ersin | Yes, it is not used in relation $\beta_2$. Ok. We have a set that includes people. And there is another set that includes meals. I want you to match the elements of these two sets but we have two conditions. The first condition is that everyone will eat a meal. And one person will not eat more than one meal. First try to do in your notebook. Then... | **ACT - Knowledge of Instructional Strategies and Representations** *Real Life Examples - Supporting* | Demonstrates teacher-selected real life examples (without endorsing the adoption of a particular example) | *The teacher gave a more spesific relation example and put in a real life example.* |
| Stud. | Do we have to write the question? | | | |
| Ersin | No, not necessary. You don't need to write the sets. I only want to see the diagram. The set of people is, "{Yaşar, Soner, Okan, Hakan}". These friends will eat something [*checking students' notebooks*]. Yes this relation is one of them. [*For another student*] Yes. Consider the conditions Ata. There are two conditions. Yes Tuğçe. Okay, nearly everyone drew similar diagrams. I will draw one. Set A consists of Yaşar,Soner,Okan and Hakan. Set B consists of meals. We can put first capitals. Kebab, Bean, Meatball, Spinach, Patato, Celery, Wrap. Yaşar likes bean. Then, could Soner also choose bean? | **ACT - Knowledge of Instructional Strategies and Representations** *Representations - Eliciting* | Wait for and listen to students' detailed descriptions of the representations in their solution methods | *The teacher gave feedbacks by checking the students' notebooks.* |
| Smartboard |   | | | |

Teacher Ersin who shows an approach of teaching by focusing on students' thoughts in general has listened to his students and encouraged them to explain their thoughts in detail. He paid attention to determining the current knowledge of his students when entering into a new concept (function concept). He provided a discussion platform within the classroom and managed the discussions well. He did not give short and strict feedbacks such as right/wrong regarding the answers received from the students. He avoided the discussion going out of context through small interventions. In addition, he made use of different representations and real-life examples.

The situation where the students have a misconception about *"If an element remains uncovered in the range set during matching, then this matching is not a function."* In Teacher Gökhan's lesson and Teacher Gökhan's approach towards this situation are shown in the below section.

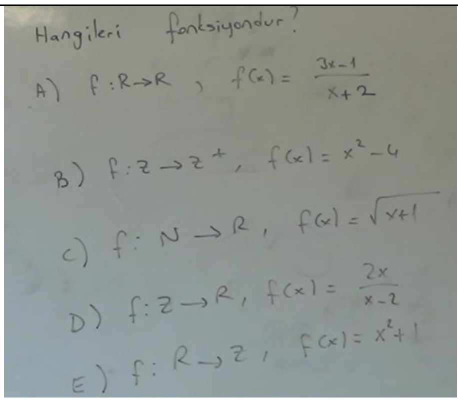
| **Source of expression** | **Expression** | **Basic Components of the framework** | **Sub-components of the framework** | **Descriptions/ Notes** |
|---|---|---|---|---|
| Board |  | **ACT - Knowledge of Instructional Strategies and Representations** *Representations - Eliciting* | Demonstrates teacher-selected representations (without endorsing the adoption of a particular representation) | *The teacher gave examples for correspondence via Venn diagrams.* |
| Gökhan | Does the second one represent a function? | **ACT - Knowledge of Instructional Strategies and Representations** *Problems - Eliciting* | Wait for and listen to students' descriptions of their solution methods | |
| Stud. (together) | No. | | | |
| Stud. 1 | Teacher, one to two …[*cannot be understood*] | | | |
| Gökhan | Okay come to the board and please tell us what you are thinking. | **ACT - Knowledge of Instructional Strategies and Representations** *Problems - Eliciting* | Encourages students to explain their responses in detail | *The teacher appreciated the student's answer and called him to the board.* |
| Stud. 1 | This one [*student is showing the diagram in the right of the board*] does not represent… | | | |
| Gökhan | Listen your friend. | | | *The teacher warned the students not to speak.* |
| Stud. 1 | In our rule each element [*pointing the* | | | |

| | | | | |
|---|---|---|---|---|
| | *conditions of being a function written on the board*] | | | |
| Gökhan | Ertuğrul is saying that this doesn't represent a function. How many of you agree with this idea? [*Few students raise their hands*] Others don't agree, true? | **ACT - Knowledge of Instructional Strategies and Representations** *Problems - Eliciting* | Shares students' solutions with all class | *The teacher asked other students' if they agree or disagree with the friend's idea.* |
| Stud. | It doesn't represent. | | | |
| Gökhan | It doesn't represent.Why? | **ACT - Knowledge of Instructional Strategies and Representations** *Problems - Eliciting* | Encourages students to explain their responses in detail | *The teacher asked "why" to deepen the student's thought.* |
| Stud. (together) | Because it isn't matched with something. | | | |
| Gökhan | That's right. | | | |
| Stud. 2 | There mustn't be any unmatched elements in the first set. | | | |
| Gökhan | What did you say? Please repeat it loudly. | **ACT - Knowledge of Instructional Strategies and Representations** *Problems - Eliciting* | Shares students' solutions with all class | *When a student told one condition of being a function, the teacher tended towards this student's thought.* |
| Stud. 2 | There mustn't be any unmatched elements in first set, but there may be in the second. | | | |
| Gökhan | Good. Actually this [*the diagram*]represents a function. What did we say? How do the elements of the first set be? | **ACT - Knowledge of Students' Thinking** *Connecting prior knowledge to new knowledge - Supporting* | Provides (students to remember) background knowledge | |
| | | **ACT - Knowledge of Students' Thinking** *Knowing students' misconceptions - Eliciting* | Uses students' misconceptions (that are determined through their explanations) for lesson's content | *The teacher realized that the students have some misconceptions, then he reminded conditions of being a function.* |
| | … | | | |
| Gökhan | Then what did we say? How many pairs does each element in the first set | **ACT - Knowledge of** | Provides (students to | |

| | | | | |
|---|---|---|---|---|
| | include? | **Students' Thinking** Connecting prior knowledge to new knowledge - Supporting | remember) background knowledge | |
| | | **ACT - Knowledge of Students' Thinking** *Knowing students' misconceptions - Eliciting* | Uses students' misconceptions (that are determined through their explanations) for lesson's content | |
| Stud. (together) | One | | | |

In this section, Teacher Gökhan recognized that there were misconceptions so he listened to the students' thoughts in order to understand the cause of these misconceptions. He did not remind the students of the conditions of being a function directly and enables the students to examine if the matching in the given Venn diagram complies with the conditions of being a function or not.

Regarding Problems-Supporting interconnection; a negative finding for Teacher Özge about the sub-component of encouraging students to analyze, compare and generalize mathematical concepts when they face a problem has been shown below as an example.

| **Source of expression** | **Expression** | **Basic Components of the framework** | **Sub-components of the framework** | **Descriptions/ Notes** |
|---|---|---|---|---|
| Board |   | | | *Teacher asked students to explain which relation represent a function.* |
| Özge | … How can I understand that there is an unmatched element in the domain set or not? If there is an element that makes the function in the A undefined, then this element will be unmatched. Namely you got an element from | **ACT - Knowledge of Instructional Strategies and Representations** | Encourages students to analyze, compare, and generalize | ***Negative:*** *There was a suitable environment in the classs for* |

| | | | | |
|---|---|---|---|---|
| | the domain set. If it doesn't have an image [*demonstrating on the Venn diagram*] then you say that it is not a function. Is there a x value that makes this undefined? | *Problems - Extending* | mathematical concepts for encountered problem | *analyzing the mathematical concepts but the teacher could not use this opportunity. She played an active role instead of engaging students in a discussion.* |
| Stud. | -2 | | | |
| Stud. | Yes. | | | |
| Özge | -2. Okay. Is -2 an element of the domain set? | | | |
| Stud. (together) | Yes. | | | |
| Özge | Then, when I substitue -2 for x and can not find an image… | | | |

This situation has been experienced in the fifth lesson of Teacher Özge when she was examining if the expressions given algebraically indicate a function or not. However, she played an active role and started to analyze the function concept by herself without waiting for the answers during this examination. This has been evaluated as a negative finding within the scope of the study.

## Discussion

The purpose of this article is to propose a conceptual framework that helps researchers examine mathematics teachers' PCK in the context of supporting students' mathematical thinking. In this study, PCK has been examined in the context of supporting mathematical thinking and has been interconnected to ACT and a new framework has been obtained. ACT framework which is composed of three components (eliciting, supporting, extending) has been interconnected to two components (knowledge of students' thinking and knowledge of instructional strategies and representations) of PCK. Instructional examples included within the framework suggested as a result of the interconnection have become the indicators regarding PCK of mathematics teachers in the context of supporting mathematical thinking.

Since the knowledge of a teacher regarding the thinking/mathematical thinking of students is considered necessary for an effective teaching, this subject has been the focus of many studies. An, Kulm & Wu (2004) suggested that knowledge of students' mathematical thinking helps teachers to enhance their own knowledge of content and curriculum, prepare lessons thoroughly, and teach mathematics effectively. They also highlighted that without knowledge of students' thinking, teaching cannot produce learning; it may instead be like "*playing piano to cows*" (a Chinese idiom) (An, Kulm & Wu, 2004). Jenkins (2010) found that the structured interview process is a way to develop prospective teachers' knowledge of students' mathematical thinking. In order to find an answer to the question "*how can teacher educators reliably assess growth in teachers' PCK?*" Norton et al. (2011) have examined school teachers' understandings of students' mathematical thinking in their studies with regard to teachers' development of PCK. For this purpose, they have developed video-based prediction assessment instruments and have experienced these. Unlike studies which examine PCK of teachers in the context of how they support students' mathematical thinking during their teaching process and which focus on students' thinking (An, Kulm & Wu, 2004; Jenkins, 2010; Kılıç, 2010, 2011; Lee, 2006; Norton, McCloskey & Hudson, 2011; Sleep & Boerst, 2012; Yeşildere-İmre & Akkoç, 2012), PCK of teachers in the context of supporting mathematical thinking has been examined within the scope of a more concrete framework in this study.

Similar to our study, Cengiz, Kline & Grant (2011) have also considered students' thinking and Mathematical Knowledge for Teaching (MKT) all together. According to the results of the study, MKT matters in the way teachers pursue student thinking. Similar to this result and in the way that validates our assumption at the beginning of the study, we have also found in this study that PCK of a teacher is important in supporting/developing students' mathematical thinking. However, unlike the study of Cengiz, Kline & Grant (2011), our study has suggested a new framework by interconnecting two frameworks (beyond examining mathematical thinking within the scope of PCK).

The suggested framework has set forth teaching components that focus on students' thinking. These components predict that the teacher pays attention to the prior knowledge, misconceptions, thoughts and questions of students, to take the individual differences into consideration, to configure the lesson in accordance with students' thoughts, to enable them to explain their thoughts, to make use of different representations, to switch between these representations and to give place to real-life examples, problems that require high-level thinking and analogies for an effective teaching. These components show similarity with the practices listed by An, Kulm and Wu (2004) for "*an effective teacher attends to students' mathematical thinking*". According to the Kulm, Capraro, Capraro, Burghardt & Ford (2001), an effective teacher attends to students' mathematical thinking: preparing instruction according to students' needs, delivering instruction consistent with students' levels of understanding, addressing students' misconceptions with specific strategies, engaging students in activities and problems that focus on important mathematical ideas, and providing opportunities for students to revise and extend their mathematical ideas (as cited in An, Kulm & Wu, 2004, p. 148).

The framework suggested in this article could help researchers in examining the teaching and PCK of mathematics teachers in the context of supporting mathematical thinking and could also enable the researchers (who focus on PCK development of teachers) to see the PCK development of teachers clearer. In fact, examples given from research (Tataroğlu Taşdan, 2014) where this framework has been used as an analytical tool in data analysis have provided the readers with an opinion about how this framework can be used in practice. Since teachers' knowledge has a complex structure by nature (Fennema & Franke, 1992), it is not easy to monitor the development of this knowledge. In the research given as an example, PCK development of mathematics teachers has been examined and it has been seen that the noted framework is beneficial for the researcher in monitoring the mathematics teachers' PCK development. When the findings of the research have been examined for each component within the theoretical scope of the study; it has been found that PCK of the mathematics teachers who participated in the context of knowledge of students' thinking component has improved most in the sub-component of determining the misconceptions of students. The real-life examples sub-component of knowledge of instructional strategies and representational have been found as the component which all the teachers have improved the most. When the findings of the same study have been considered within the scope of ACT; it has been found that the participant mathematics teachers are more successful in eliciting and supporting steps of the model. In their studies, Fraivillig, Murphy and Fuson (1999) have also found that teachers are more successful in the supporting steps and less successful in eliciting and extending steps. They have indicated the source of this difference as the differences in pedagogical skills of teachers required for eliciting, supporting and extending. The source of this difference between the findings of the two studies may be the differences in teachers' education, curriculum etc. of different countries.

## Conclusion

It is thought that this study has made a contribution to the field by examining PCK in the context of supporting mathematical thinking and showing through which practices could a mathematics teacher be able to support the students' mathematical thinking throughout his/her instruction. The suggested framework is a useful tool for the researchers and teacher educators who are dealing with teachers' knowledge focusing on students' mathematical thinking and a guide for the teachers. As Fraivillig, Murphy and Fuson (1999) have suggested for ACT, this developed framework can serve as a beneficial pedagogical tool in pre-service and in-service teacher education. In addition, the framework is an analytical tool that can be used for monitoring mathematics teachers' PCK development.

The limitations of ACT Model and PCK which form the basis of this framework are also the limitations of the framework suggested in this study. Fraivillig (2001) has evaluated ACT model as "*Although eliciting, supporting, and extending describe elements of effective instruction, the art of teaching is much too complex to be captured by these three components*". The multi-dimensional structure of PCK and the complex structure of teaching make it difficult to define the teaching process through explicit components. For this reason, it should be considered that the framework suggested in this study may not work at all times. Besides, the framework has focused on only two components of PCK (knowledge of students' thinking and knowledge of instructional strategies and representations). Examining other components of PCK within the scope of this framework and the effects of this on the mathematics teaching process focusing on students' mathematical thinking can be examined in the further studies.

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