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CONCEPTUALIZING PRE-SERVICE MATHEMATICS TEACHERS' RESPONDING TO STUDENTS' IDEAS WHILE TEACHING LIMIT CONCEPT

Semiha Kula Ünver[1], Esra Bukova-Güzel[2i]
[1,2]PhD, Dokuz Eylul University, Faculty of Buca Education,
Departments of Mathematics Education, Buca, İzmir, Turkey

Abstract:
The purpose of this study was to conceptualize pre-service mathematics teachers' responding to students' ideas, one of the codes of Contingency unit of Knowledge Quartet, while teaching limit concept. The participants were four pre-service secondary mathematics teachers. The data were obtained from the lesson plans, the video records of the participants' lessons, and the semi-structured interviews. When the data were analysed, the seven sub-codes of the pre-service teachers' responding to students' ideas were determined. These sub-codes were named as (a) repeating students' ideas, (b) approving students' ideas, (c) explaining and expanding students' ideas, (d) answering students' questions, (e) asking how students' reached their ideas, (f) correcting mistakes in students' ideas, and (g) ignoring students' ideas. It is thought that these sub-codes would be helpful to examine pre-service mathematics teachers' responding to students' ideas in a detail way.

Keywords: contingency; knowledge quartet; limit concept; pre-service mathematics teachers; responding to students' ideas

1. Introduction

The Knowledge Quartet (KQ) is a framework for the observation, analysis and development of mathematics teaching, with a focus on teachers' subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (Rowland, Huckstep, &

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[i] Correspondence: Dr. Esra BUKOVA GÜZEL, Associate Professor,
E-mail: esra.bukova@gmail.com, esra.bukova@deu.edu.tr.

Thwaites, 2005; Rowland, Turner, Thwaites, & Huckstep, 2009). The KQ has four dimensions-called Foundation, Transformation, Connection, and Contingency-each of which is associated with several codes (Rowland, Huckstep, & Thwaites, 2005). Foundation, the first unit, involves theoretical background about SMK and PCK, as well as beliefs regarding mathematics and mathematics teaching. Transformation comprises the ways in which knowledge can be transmitted clearly by teachers to learners, the use of examples and selection of procedures to form concepts, and the choice of illustrations and representations. Connection involves decisions about sequencing subjects or lessons, associating lessons with previous lessons and with students' knowledge, associating procedures with concepts, and anticipating and carefully sequencing the introduction of complex ideas in the lesson. Contingency, the focus of this study and the last unit, involves unplanned examples in lessons, such as students' unexpected ideas, deviation from the lesson agenda in response to an unplanned opportunity.

Rowland, Thwaites and Jared (2011) suggested that the consideration of contingency -including its possible triggers, consequences of such triggers, and demands on teachers' various knowledge resources- had an important but, as yet unrecognized, place in mathematics teacher education. Mathematics teachers may not predict contingent moments before they happen as well as student teachers. Although the teachers plan their teaching in accordance with their insights of possible students' responses their experiences, knowledge of students, and content knowledge (Ball, Thames, & Phelps, 2008), they may not be able to predict each student's response (Turner, 2009). Schoenfeld (1998) handled "lesson image" as contingency. He stated that teacher's lesson image included knowledge of students and how they may react in the planned lesson, what students are likely to be confused about, and how the teacher overcome these difficulties (Rowland, 2010).

Contingency deals with the situations that cannot be planned for before the lesson (Rowland et al. 2009; Rowland, Huckstep, & Thwaites 2005). The idea that while most situations in the classroom can be planned, some cannot, prompted the researchers to generate this unit. Contingency also covers the ways in which teachers respond to unplanned instances in a lesson, which often tests their ability to 'think on their feet' (Rowland, Huckstep, & Thwaites, 2005). Contingency is composed of four codes named as deviation from lesson agenda, teacher insight, responding to the (un)availability of tools and resources, and responding to student's ideas. According to us, the 'responding to student's ideas' becomes more prominent in comparison with other codes. As responding to student's ideas may lead to deviate from the agenda and shape the teacher's insight, this study focused on f responding to students' ideas.

2. Responding to Students' Ideas

Turner (2009) suggested that it was possible to ensure more meaningful teaching by responding to students' ideas. Additionally, teachers acquired more information about the nature of students' knowledge when they articulated an idea not to be predicted (Thwaites, Huckstep, & Rowland 2005). To put aside such opportunities, or simply to ignore or dismiss ideas as 'wrong', can be meant as a lack of interest of students' ideas (Rowland et al. 2009).

Turner (2009) used the KQ as a framework for observing, reflecting and discussing the teaching practices of 12 pre-service mathematics teachers starting their practices during a period of four years. She indicated that the participants exhibited improvements in terms of responding to students' ideas by the end of the four-year period. They effectively dealt with to students' errors and used unplanned resources at the end of this four-year. They became more proficient on understanding and discussing of student's methods and ideas and basing their teaching by the way; they also began to adopt a more inquiry-based approach to their mathematics teaching.

Other researchers (Ball 2003; Ball & Sleep 2007; Empson & Jacobs 2008; Even & Tirosh 1995; Graeber, 1999; Lloyd & Wilson 1998; Marks 1990; Sleep & Ball 2009; Tirosh, Even, & Robinson 1998; Van der Valk & Broekman 1999) also emphasized the importance of responding to students' ideas. For instance, Ball (2003) pointed out the importance of controlling discussion and evaluating students' verbal and written responses. Schoenfeld (2006) emphasized that when an unexpected incident happened in the lesson, the teachers would review their goals at that moment. The students' ideas such as comments, questions and answers may create unexpected situations for the teachers, but it is considered important for them to take advantage of such opportunities to develop their teaching. For instance, Rowland, Thwaites and Jared (2011) examined Bishop's (2001) anecdote about a class of 9- and 10-year-olds who were asked to give a fraction between ½ and ¾. One girl answered ⅔, explaining "because 2 is between 1 and 3, and 3, the denominator, lies between 2 and 4. They indicated that the girl's answer was probably not one that the teacher had expected, and the teacher could ignore or effectively dismiss the girl's proposal or alternatively take it seriously by trying to understand her reasoning.

A teacher's ability to give appropriate responses to student's ideas requires primarily to listen the student's ideas and to understand them. When the teachers listen to their students, they can shape their teaching by means of students' ideas (Wicks & Janes, 2006). Despite the fact that the need to utilize the students' ideas, the researchers showed that listening to students' thinking was hard work especially when students' ideas sound and look different from teachers' expectations (Ball, 1993; Morrow, 1998; Wallach & Even, 2005 cited in Suurtamm & Vézina, 2010). Additionally, sometimes teachers may ignore the students' ideas even if their ideas are appropriate for the goals or able to contribute to the subject. For instance, teachers tended to minimize or dismiss the students' ideas when they were not listening to them or when they do not understand such thoughts (Cobb, 1988 cited in Davies & Walker, 2007). It is important to listen and to understand students for responding their thoughts. Especially, effective listening is also significant for understanding students' unexpected ideas and how these ideas are formed. So, teachers should try to consider their students' thinking and respond them effectively.

In this study, the pre-service mathematics teachers' responding to students' ideas were examined in the context of teaching limit concept. The study focuses on limit concept because each concept in the Calculus depends on limit and it is suggested to teach the concepts of continuity, derivative and integral by depending on the limit concept in National Mathematics Curriculum in Turkey. In the next part, we discussed important points of the limit concept.

3. Limit Concept

Even though the limit concept is very important, students and teachers have difficulty in learning or teaching this concept (Bukova, 2006). Researches (Davis & Vinner, 1986; Orton, 1983; Williams, 1991) showed that students had some misconceptions about limit such as seeing limit as a boundary, an approximate value, an infinite process and a value (Blaisdell, 2012). Additionally, Thabane (1998) and Laridon (1992) expressed that students thought the value of the function as similar to the value of the limit (Jordaan, 2005). Kula (2011) also stated that students tended to generalize the property of polynomial functions for finding limit to all types of functions. In our country, the limit concept was introduced in the secondary school and limit concept is taught intuitively. It could be thought that the students had more unexpected ideas and more questions on limit. So, the pre-service teachers' may generally meet the contingent moments while teaching limit. Additionally, having knowledge of students' ideas, errors, misconceptions, and difficulties may reduce these moments. It was known that pre-service teachers were not very well-placed to anticipate contingent moments, because they did not reflect their theoretical knowledge about SMK and PCK in their teaching (Rowland, Thwaites, & Jared 2011).

In this study, the mathematical content focus was the *limit concept*; the purpose of the study was to conceptualize the secondary pre-service mathematics teachers' responding to students' ideas occurred in contingent moments. For this purpose, the research question was determined as "How to conceptualize the pre-service mathematics teachers' responding to students' ideas occurred in contingent moments while teaching limit concept?".

4. Methods

This was a qualitative case study which aimed to conceptualize the pre-service mathematics teachers' responding to students' ideas and four cases were chosen to fill code named responding to students' ideas, to extend the emerging code.

4.1 Participants

The participants in the research were four (three females and one male) secondary pre-service mathematics teachers. The participants were senior undergraduate students enrolled in a secondary mathematics teacher education program. The secondary pre-service mathematics teachers enrolled in a five-year program in Turkey. During the first three years, the pre-service mathematics teachers' took the courses such as Calculus, Analytic Geometry, Discrete Mathematics, Differential Equations, Algebra, Complex Analysis, Topology, Probability and Statistics, etc. In addition to these courses, they also took courses such as Mathematical Modeling, Mathematical Problem Solving, Mathematics and Art, Mathematics and Games, History of Mathematics, Mathematical Applications with Computers, Mathematical Thinking, and New Approaches in Mathematics. During the last two years, for preparing them to teach mathematics, they took courses regarding both general pedagogical knowledge and PCK such as Introduction to Educational Sciences, Curriculum Development, Assessment and Evaluation, Classroom Management, Guidance, Teaching Methods in Mathematics, Examination of Mathematics Textbooks. Apart from this, in the last three semesters, there were courses related to school-based placement named School Experience I- II and Teaching Practice. During this education process, the participants had the chance to improve their knowledge about the limit in the Calculus I-II courses. At the same time, in Teaching Methods in Mathematics I-II they had also chance to learn how to teach this concept.

The participants took part in the research voluntarily and chose their pseudonym such as Deniz, Umay, Caner and Alev. The participants had almost completed the ninth semester in the program; therefore, they completed their course, except the last semester. They realized their lessons about the limit within the scope of the course School Experience II. Before teaching limit, they prepared lesson plans for four lessons. In their class, there were approximately 11-15 students.

According to the mathematics curriculum in Turkey, the students were introduced to the limit concept for the first time in their senior year in the secondary school. Before teaching limit, the students thought the topics such as relation-functions, the domain and range of functions, algebraic and transcendental functions (such as trigonometric functions, logarithmic functions, polynomials functions etc.), and absolute-value functions at different grades. In the national mathematics curriculum, the limit concept was not defined formally and it was given as the form of limit which was described intuitively in the literature (Bergthold, 1999; Cornu, 1991).

4.2 Data Collection

Data were obtained from the participants' lesson plans for the limit concept, video records of their sixteen lessons (four lessons per participant), and the audio records of the semi-structured interviews (seven interviews per participant).

The participants prepared for four lesson plans before teaching limit. These plans were taken from the participants to determine whether the participants carried out their lesson as planned. So, it was tried to handle whether the participants adopted/revised/changed their teaching in the direction of students' ideas and the participants' approaches performed in the contingent moments. With the lesson plans, we determined whether the cases were occurred, the participants changed their teaching and how they changed their teachings by use of students' unexpected ideas. The participants' lessons were recorded to examine the participants' responding to students' ideas during their teaching. By these video records, it was determined the unexpected events which they did not include in their lesson plans

The forms of semi-structured interviews were individually prepared for each participant by considering their lesson plans and video records of their lessons. The focus of these interviews was any relevant occurrences of contingency in lessons. We interviewed them to understand their awareness about the contingent moment. We also determined why this moment was occurred and finally we asked them to interpret this moment. These interviews were made to investigate how the participants responded to students' ideas and why they were response in this way. The interviews were realized before and after the lessons.

4.3 Data Analysis

The video records of the participants were transcribed verbatim after descriptive synopses had been prepared. The descriptive synopses of the lessons were prepared to use in the interviews and contained the important events of what happened in the lessons. They were used to determine the questions which the participants would be asked in the interviews after their lessons. The participants' lessons were watched two or three times. For verbatim transcriptions, a format was determined. It included the words of the participants and their students, the screenshots of the presentations and videos of participants, and the statements that the participants and the students wrote on the board. The voice records of the interviews were also transcribed verbatim. These transcripts were used to support the findings related to responding to students' ideas.

When the data were analyzed, the researchers firstly familiarized with the data at hand while transcribing the video of the lessons and secondly with re-watching and relistening them. The videotaped lessons aspects of participants' actions in the classroom related to responses to students' ideas were identified. The transcripts of videos were analyzed, started by the first lesson's transcript, line-by-line via open coding with a goal to generate categories and their properties (Corbin & Strauss, 1990). A new sub-code was generated whenever a pattern in the data were identified. Throughout the analysis process, the sub-codes to each other to observe similarities and differences between them were continuously compared. Also how the sub-codes related to each other were tried to identify.

While coding process, it was constantly compared the incident with previously coded incidents in the same and different categories, as well as continually compared the generated categories to each other to understand the relations between them. While generating sub-codes, the transcriptions of each lesson were reviewed many times. They were examined in terms of whether the teachers responded to the students' ideas or not; the cases in which the participants did or did not show interest in these ideas etc. were determined, and sub-codes relating to these cases were formed and appropriately named reflecting the context.

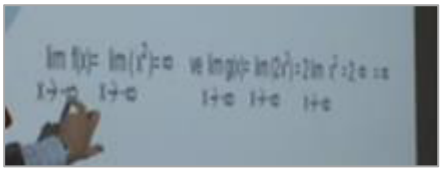
5. Results

At the end of the data analysis, when responding to students' ideas, the pre-service mathematics teachers used the seven different ways of responses. These sub-codes of responding to students' ideas were: (a) repeating students' ideas, (b) approving students' ideas, (c) explaining and expanding students' ideas, (d) answering students' questions, (e) asking how students' reached their ideas, (f) correcting mistakes in students' ideas, and (g) ignoring students' ideas.

5.1 Repeating students' ideas

In this sub-code, the students' ideas such as answers of a question, inferences, interpretations, and explanations were repeated verbatim. The students' ideas thought they were true were generally repeated but rarely the students' incorrect ideas, misinterpretations, incorrect answers and incorrect inferences related to the mathematical concepts were also repeated. While repeating students' ideas, the participants did not add anything and did not ask questions related to what they think. The participants tried to give the impression that they were interested in what was said. But in some cases, they repeated the inadequate or incorrect explanations. The inadequate or incorrect explanations repeated sometimes for attracting students' attention providing opportunities for students to think one more time. However, sometimes the participants repeated students' ideas without evaluating whether the ideas are true or not. For instance, Umay repeated the one student's expressions related to the "increasing in the negative direction" which were mathematically wrong in her fourth lesson.

Slide:



$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^2) = \infty \text{ and } \lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} (2x^2) = 2 \lim_{x\to-\infty} x^2 = 2\infty = \infty$$

Umay: What is that? It means x decreases without bound.

Student: Hmm, yes. It is increasing in the negative direction.

Umay: Yes, we can say that it is increasing in the negative direction. What can we say about the function?

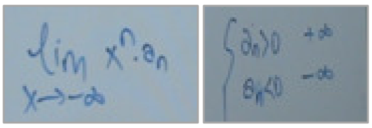
Umay repeated her student's wrong expression about minus infinity stating as it is increasing in the negative direction without paying attention to the mathematical correctness. Actually, her first expression about minus infinity was mathematically true. But she made a mistake while repeating the student's ideas. Her approach could cause to problems about sequencing in real numbers. In her fourth lesson's interview, she stated the following:

Actually, now, when you asked, I realized the mistake of the student's expressions. Sometimes, I spontaneously repeated my students' responses without evaluating. (Umay-Interview after the fourth lesson)

5.2 Approving students' ideas

Another way to responding students' ideas was approving students' expressions. The students' ideas were approved to show them that their thoughts were correct, they were listened or to encourage them to continue their actions. The terms such as "yes", "ok", and "that's true" were commonly used to approve students' ideas. Sometimes the approvals were done through mimics or body language (such as nod). Umay asked her students' what the $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ polynomial function's limit was at $\pm\infty$. She called one student to the board and followed him while solving the problem. She approved his solution steps and showed that she was following him and also encouraged him to continue as seen in her fourth lesson.

Board:



Student: Here when n is an odd number a_n can be positive or negative or when n is an even number a_n can be positive or negative.

Umay: Yes (*She shakes her head*).

Student: $if\ n$ is odd and a_n is positive $\lim_{x \to -\infty} x^n . a_n$ will be minus infinity. If a_n is negative then it will be plus infinity.

Umay: Yes.

Umay approved her student step by step while he was explaining his ideas. In the fourth lesson's interview, she explained the reason of approving as follows:

Sometimes, the student on the board wrote slowly because she/he did not be sure about their solution. I mean, whether what he/she do is true or not. And doing that, continuously they turned back and looked at me and her/his friends. For that moment, I feel that they needed approving. And, when I approved them, they relaxed and were more willing to continue. Actually, I tried to approve them whenever I felt. Not only their solutions on the board but also anywhere they expressed their thoughts. (Umay-Interview after the fourth lesson)

5.3 Explaining and expanding students' ideas

This sub-code was on explaining and expanding on students' ideas. By explaining and expanding the students' ideas teachers enabled students understand what it was thought, why it was thought this way, how this idea came up, what this idea was related with, and what other ideas lead to this idea, how this idea could be enlarged by considering different conditions etc. So, other students could be understood one student's ideas. Also, this encouraged other students think based on the idea. By explaining students' ideas, some possible missing points for other students could also be fixed. When the student on the board did not explain his/her solutions or when it was intended to explain the solutions step by step, the participants explained these solutions to whole class. They also expanded the students' solutions and ideas by giving related properties and mathematical rules or by discussing underlying reasons of these solutions.

Alev gave students a piecewise function and its graph. She asked the students what they thought about the limit of this function when $x = 4$. She explained the student's comment to others by expanding on it.

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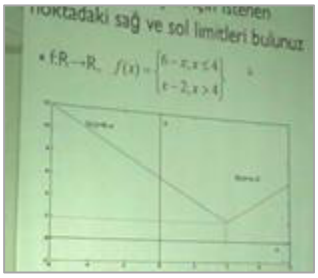


Alev: What can be said here? About the limit when x is going to 4

Student: There is limit if we find the same value when approached from left and right.

Alev: Yes. We use different functions' rules while investigating from the left and right hand limit, don't we? Because our function is a piecewise function. So, we have to select the valid rule of a function according to the given range.

5.4 Answering students' questions

The students asked questions while they were expressing what they thought. This sub-code covered the responses given to the students' questions. Sometimes the teacher gave answer by himself/herself and sometimes the teacher enabled students find the answer by themselves. Several directive questions were asked for students to reach the answer and students were given a chance to review their opinions and improve them. In addition to this, some answers were given based on mathematical rules. On the other hand, sometimes the participants answered their students' questions without reasoning because it was intended to quickly.

Deniz introduced the properties of the limit concept in her second lesson. She first taught the properties of addition, subtraction and multiplication. Then, one student asked her whether division had a similar characteristic or not. Deniz paid attention to her student's question and tried to answer it.

Slide: 6) $g(x) \neq 0$ *ve* $\lim_{x\to a} g(x) \neq 0$ *için* $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Student: Does the same thing happen in division?

Deniz: Let's see if it does. This time we say $\frac{1}{f(x)}$ instead of $f(x)$ here. That is similar to multiplication. So it is similar to multiplication. If we multiply $\frac{1}{g(x)}$ with f(x) we get this equation. Ok then why I gave ($g(x) \neq 0$ *and* $\lim_{x\to a} g(x) \neq 0$) condition at the beginning?

Student: Because it is indefinable.

Deniz: Because if these conditions are not provided division is indefinable.

In another part of her lesson, Deniz asked her students to find the limit of the functions at $x = 2$ by using the given two different functions' graphs (see Figure 1). Upon this questions, students said that the functions had no limit at x=2 according to their graphs. However, Deniz said both functions had a limit at this point and the limit value was 1. When they were not convinced about $\lim_{x\to 2} f(x) = 1$ and asked the reason of it, she tried to convince them by saying that "this was a rule, you could also find it in your books".

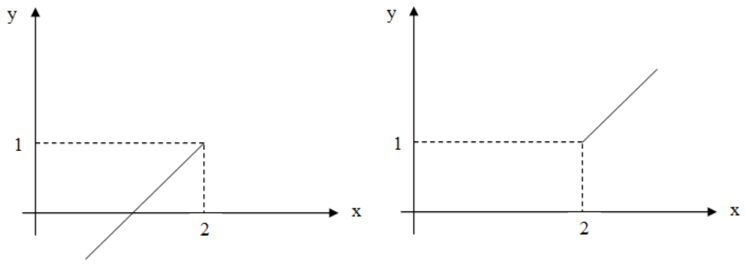


Figure 1: Deniz's examples of two graphs

In the interview, this excerpt of her lesson was showed and she was asked what she thought about her explanations. She answered the question as follows:

As I said before, this is a case in which I don't believe myself. I think that there is not a limit only from the right or only from the left. I mean I think that it cannot be expressed in this way. Because it contradicts our earlier description of limit. However, I did not know how to express this. Now if I had expressed my ideas, the students would have been more confused. If I said "I think that is it", they would also think in the same way and this might lead them to make mistakes. So I said this. The books express it in this way. It is accepted in this way. So it is in this way. It was more like "go and learn". It was not right but I did not know how to express it in another way because as I said, I thought there was no such thing, and so I had this situation as I did not know how to explain. (Deniz-Interview after the third lesson)

Caner, in his fourth lesson, when a student asked "is infinity a number, or is it the whole number line?", he tried to answer.

Student: Is the plus infinity used as a number or is it the whole number line?

Caner: It is not the whole number line. Look now what am I doing? I am drawing a graph… 1 divided by x function, if we draw this here.

Board:

Caner: Now we will draw this graph, too. I drew this. As x tends to zero, what is the limit of this?

Student:	To infinity.
Caner:	x tends to zero from the right. Then, what is the limit?
Students:	To infinity.
Caner:	Does it go to plus infinity? Did we search the whole line? No, what did we do then? We think that it will go to the edge of the line, right?
Student:	Yes.
Caner:	Or what if I search for the limit at infinity on the x line. What does it equal to?
Student:	Limit? Or zero?
Caner:	If I search the limit of this function at infinity.
Student:	Infinity.
Student:	What does it equal to?
Student:	To zero.
Caner:	Does it equal to? Now if I take a point here, will it come here? What if I take here?
Student:	Tends to zero.
Caner:	More. More to here, to more and more here to what am I tending?
Student:	To zero.
Caner:	I am tending to zero, right? What I am doing here, I am using infinity as a number, right? Actually, infinity is not any number.

He stated that the students' questions were important for developing lessons and understanding what they really thought.

This question was a sign for me which showed that the students still had a question mark in their minds about the concept of infinity. The lesson was interrupted by this question. I moved away from the lesson plan and tried to answer their question. But, I noticed the students' questions because they give me some clues for their ideas. I can use them for conducting my lessons. And then, I try to answer in an appropriate way. (Caner-Interview after the fourth lesson)

Alev was aware of her students' questions, but was not able to give correct explanations because of her lack of knowledge about indeterminate forms of limit. For example one student asked her "Is the sum of infinity and infinity is again infinity?". Her answer was "I do not know. It should not be." Alev noted that her content knowledge was not enough to deal with the student's questions. She also stated that sometimes she answered the students' questions without reasoning.

To tell the truth, I do not think that my knowledge is enough. I have many deficiencies. Actually, I have been doing something to make up for my deficiencies; I read things about this subject or I try to come across with different questions etc. because I think that I have many deficiencies both in my subject matter knowledge and teaching. (Alev-Interview before the lesson).

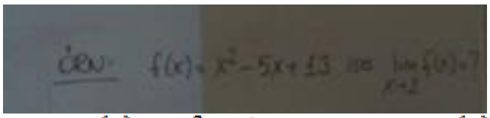
I had difficulty in terms of knowledge about limit. As I had not solved many examples, I had to think about the subject for a time even though I knew the subject. I tried to be fast in thinking. Sometimes I had to answer before thinking. I mean I felt that I had to answer. Not to think about it for a long time in front of the students. I had this kind of difficulty. (Alev-General interview about the lessons)

5.5 Asking how students' reached their ideas

This sub-code was included to reveal how students reached their ideas. When the participants interested in their students' reasoning without only evaluating their results this sub-code was revealed. To reveal underlying reasons of students' ideas, the participants asked students their reasons. It enabled to be informed about students' obstacles, misconceptions and mistakes. At the same time, sharing students' ideas with their classmates' enabled students understand each other's thought, discovered different thought patterns and fixed their incorrect thought. There were examples of teachers asking students how they reached their answers or why they made certain comments. Discussion about such things could help students express themselves better, other students could also benefit and the teacher understood their reasoning better.

In the second lesson, Deniz asked one of her students to explain how to find the limit. So, she tried to understand the underlying reasons of her student's response.

Board:



Example: $f(x) = x^2 - 5x + 13 \Rightarrow \lim_{x\to 2} f(x) = ?$

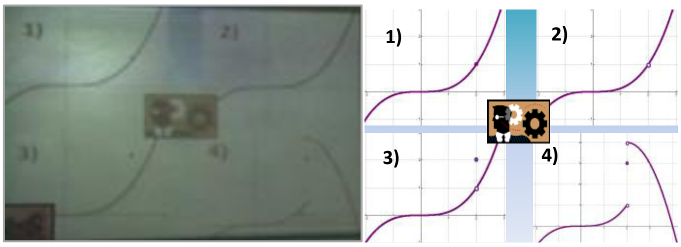
Deniz: Seven, why seven?
Student: It tends to two.
Deniz: Why did you put two?
Student: As it tends to two.
Deniz: But can we put two every time? Maybe the function is not defined there.
Student: Polynomial.
Deniz: Yes as f function is a polynomial function, we can directly put the x value in the functions. You remember this rule, right?
Student: Already I did it.

Deniz: Do you use this rule in other functions?

Student: Not, not in all functions, but it is polynomial function. It works in polynomial.

In another excerpt from Deniz's lesson, she discovered that they held the misconception of *a limit being a bound that cannot be attained* (Szydlik, 2000; Williams, 1989, 2001 cited in Kula 2011). In group work, Deniz asked the students to examine whether the function which graphs were given could be limited at $x = 2$ or not. Her students thought that if the function was defined at $x = a$ it could not be limited on that.

Slide:



Student: In the first three graphs, if we tend to two from the left or from the right, we tend in the same number.

Deniz: Okay.

Student: But as the number two takes value in the first, third and fourth graphs, it is only on two.

Deniz: Only?

Student: On two.

Deniz: Here (*showing the second graph.*) you say there is a limit. I do not understand. Can you explain the reason one more time?

Student: Because as in the first, third and fourth graphs the function is defined at $x = 2$ and takes value, so it is not limit.

Deniz: As it takes a value more than one.

Student: As it takes value. As it is defined.

Student: When it comes to two in the others there is a value but in the second graph when it comes to two, the function is undefined.

Deniz Okay (showing the first graph) here?

Student: When we tend to two there, there is also a value.

Deniz: Is there anything you want to add? You can say something.

Student: Now we said in the first one, it can never come to 2 even though we make it continue to the infinity. For this reason I think there is not a limit there. When we come to two, there is a limit. When we tend to two from the left, one goes to one. When we approach from the right, it again goes to one. There is a limit there.

In interview, Deniz stated that she realized her students' misconceptions by asking their reasoning and explained that she did not expect her students to develop such kind of idea.

To what do I attribute the students reaching the solution that there was no such limit here? Actually, I was not expecting that they would reach this result… What did I do to remove it? I mean to remove it first I tried to hear their ideas by showing them more than one graph. (Deniz-Interview after the first lesson)

5.6 Correcting mistakes in students' ideas

This sub-code dealt with correcting the mistakes in students' ideas. These mistakes took place in students' solutions, answers or comments. Students' solution processes were followed and their uses of mathematical language were paid attention to and their mistakes in both solution steps and in the use of mathematical language were corrected. While students express interesting solution ways, they could make mistakes. By giving feedback to students about their mistakes, misconceptions and obstacles etc., the participants could help them to discover their mistakes. Sometimes students' mistakes were corrected immediately and sometimes further questions were asked to the students that lead them to correct their own mistakes. Mistakes that were noticed in students' ideas through the *asking how students reached their ideas* were corrected in this sub-code.

As seen in the -*asking how students reached their ideas*-in Deniz's lesson, when her students gave wrong explanations, Deniz questioned their reasoning. This led her to understand the students' misconception, and in her second lesson to overcome the misconception she gave examples from daily life.

Deniz:	Okay. Now I will give you an example, from daily life. This may help you a little. Umm let's assume that there is a girl, okay?
Student:	Is she beautiful?
Deniz	Now her beauty will become clear. Two boys like her and they approach her. Okay?
Student:	(*Not understandable*)
Student:	Does the girl approach both of them?
Deniz:	The girl doesn't approach either of them. She doesn't know them. Both of these boys approach her. But the girl likes another boy.

ooo

Deniz:	Now there are functions and we tend to. Now I am explaining the point of this example. We approach to a point from the left and right. But we don't have to be the thing that we tend to.
Student:	We don't have to.

Deniz: We don't. Yes. But we can.

Student: Then 1 can.

Deniz: Umm, does 1 fit this situation?

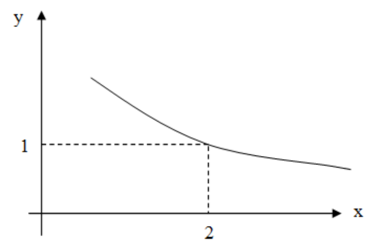
Students: It does.

Deniz: It does. We said we don't have to. (*Pointing the second graph*) does this fit?

Student: Yes.

Deniz corrected the students' mistakes, as demonstrated in the extract in her second lesson. Deniz gave the six different functions' graphs to her students and asked them to examine the limits of the functions at $x = 2$. One of her student said that the function represented in the 4th graph has not limit at $x = 2$. When she questioned her student's reasoning, she understood the reason of mistake. Her student reached this reason because he thought that only the function represented in 4th graph was decreasing.

Slide:



Deniz: Why did you say that there is no limit in the fourth graph?

Student: Or third and fifth. As the graphs are similar. That one is different.

Deniz: How different? What if the fourth graph was like this. What would happen if it were like this? (*as the function which represents the 4th graph is decreasing, she draws a function on the board to make the students understand that the mistake is related to increasing-decreasing functions*)

Student: No, not from that.

Deniz: Why? We changed the shape, is there a problem in the function?

Student: It looks like it's decreasing.

Deniz: Now if our values of x tends to two, wouldn't we tend like this from this curve?

Student: But what about after? It looks like it's moving away from two.

Deniz: But we are coming to two. Look here, if the values of x come to two…

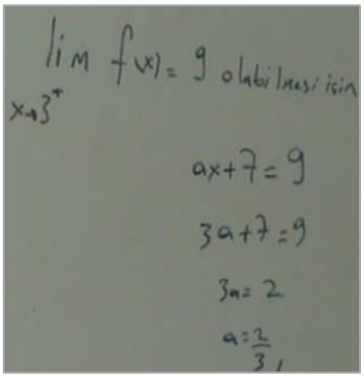
Student: Okay. Okay then.

Deniz: …if it tends to two this way, this also tends to two, right?

Student: Then fourth is okay also.

Caner corrected his students' mistakes in their written mathematical expressions on the board.

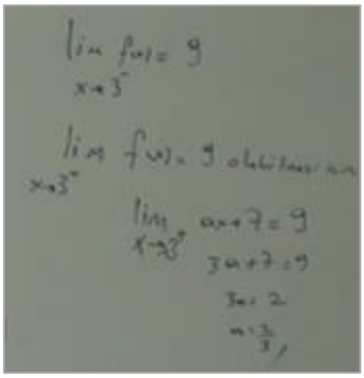
Board:



(*Students' writing*)

Caner: Okay. Thank you. (*The student did not use the proper notation to show that the limit of* $ax + 7$ *is equal to* 9 *while tending to* x 3 *from the right*). Look here (*showing* $ax + 7 = 9$). There is something missing here. We are adding the limit while tending to x 3 from the right. (*He corrects students' mistakes as follows*)

Board:



$\lim_{x\to 3^-} f(x) = 9$

$\lim_{x\to 3^+} f(x) = 9$ olabilmesi için

$\lim_{x\to 3^+} ax + 7 = 9$

$3a + 7 = 9$

$3a = 2$

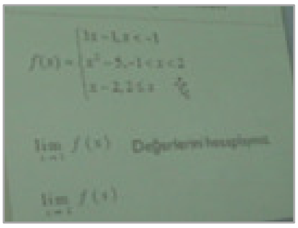
$a = \frac{2}{3}$

5.7 Ignoring students' ideas

This sub-code focused on the situations in which students' ideas were not paid attention. The students' ideas, answers and comments were sometimes ignored. Ignoring ideas occurred sometimes the students' ideas were out of the context of the subject or sometimes the teacher had not knowledge related to ideas. Sometimes the students' ideas were ignored without specifying them as true or false and sometimes, the incorrect responses were ignored and it was not questioned the reason why the students thought in this way.

Alev sometimes reacted to her students' questions or comments negatively (e.g. "no, it is not!") and then, instead of asking their reasoning, she immediately gave the answer which she wanted to hear herself. She also pretended on occasion not to hear answers her students gave, ignoring them completely. As can be seen in her third lesson, before a consensus was reached about the answer to a question, she moved on to another one.

Slide:



$$f(x)=\begin{cases}3x-1, & x<-1\\ x^2-5, & -1<x<2\\ x-2, & 2\leq x\end{cases}$$

Calculate the values of $\lim_{x\to 1} f(x)$, $\lim_{x\to 2} f(x$

Alev: For example, calculate these values for this function. Take your pencils and write.

Student: Not in 1.

Student: 1 in 2.

Student: Not in 1.

Student: Not in 1.

Student: -4.

Student: Yes, not in 1.

Student: -4 in 1. Not in 2.

Student: Not in 2.

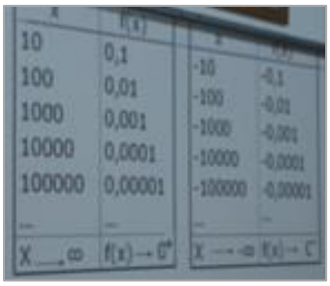
Student: Yes, not in 2.

Alev: (*After 54 seconds, Alev moved to another slide without showing any interest in these comments, even though the students couldn't agree on an answer*) Now, what can we say about piecewise-defined function?

The students asked Umay whether there was a relationship between recurring decimals and limit concept. She replied to these questions superficially, without explanation or encouragement. She did not give a clear answer, but tried to ignore them. Umay did not always pay attention while the students were discussing a question. One such discussion was about whether 1 divided by 0 was infinity or undefined: Umay responded by simply stating that they were considering 1 divided by 0, and did not attempt an explanation.

Student: Can I ask something? Is "1 divided by 0" infinite?

Side:



x	f(x)
10	0,1
100	0,01
1000	0,001
10000	0,0001
100000	0,00001
...	...
$X \to \infty$	$f(x) \to 0^+$

x	f(x)
-10	-0,1
-100	-0,01
-1000	-0,001
-10000	-0,0001
-100000	-0,00001
...	...
$X \to -\infty$	$f(x) \to 0^-$

Student: Isn't it undefined?
Student: Infinity.
Student: One divided by zero.
Student: Undefined.
Student: Undefined.
Student: Undefined?
Student: If you approach from the right, it is infinity, though.
Umay: We are directly talking about one divided by zero. Yes, let's continue, friends (*He continued before answering the question "is 1 divided by 0 infinite?"*). What did we do later? First we tended to zero. After a time, we tended to infinity. Because we learned the concept of infinity. How did we tend?

In the post-lesson interview, Umay explained why she did not give a clear explanation to this question as follows:

At that moment, I could not comprehend students' conversations. I hear their voices as a humming noise. It has evolved besides me. Actually, they started to talk to each other. And I did not add something like that to my lesson plan. This was unexpected. I skipped this thinking that we would talk about it later. I mean I put it off. For this reason, I didn't give any instruction or explanation. (Umay-Interview after the fourth lesson)

6. Discussion

In the study conceptualized the four pre-service mathematics teachers' responses to their students' ideas, it was constituted seven sub-codes. These sub-codes were named as (a) repeating students' ideas, (b) approving students' ideas, (c) explaining and expanding students' ideas, (d) answering students' questions, (e) asking how students' reached their ideas, (f) correcting mistakes in students' ideas, and (g) ignoring students' ideas.

The pre-service mathematics teachers repeated students' ideas, and approved the things they said, to show that they were listened. While repeating students' ideas, the pre-service mathematics teachers did not add anything and did not ask questions related to what they said. Approvals were sometimes done through terms such as "yes", "ok", and "that's true" or sometimes with mimics or body language. It could be important that teachers approve their students' ideas for students to state their ideas easily and willingly. Bass and Ball (2000) expressed that pedagogical content knowledge included the ability to guide the course of mathematics discussion in determining whose comments to include, explore, expand on, when to push students to continue, what explanations to provide (Leavit, 2008).

When the pre-service mathematics teachers were interested in what students were saying and doing, they would explain and expand on what they saw and heard, thus they contributed to the understanding of others. The sub-code named as explaining and expanding students' ideas in this study was encountered as explaining and guiding explanations in the central tasks of mathematical knowledge for teaching proposed by Ball and Sleep (2007). Wicks and Janes (2006) named the sub-code of the explaining and expanding students' ideas as clarifying in their studies. According to them teachers made clarifying to explain their students' ideas and clarifying seemed important for re-stating of the thoughts for the benefit of other students in the class and processing what they were saying as well as making connections.

Students sometimes asked questions while expressing what they thought. The sub-code named answering students' questions covered the responses given to students' questions. The pre-service mathematics teachers sometimes gave the correct answer or sometimes enabled the students to find the answer by themselves.

The sub-code named asking how students' ideas were reached their ideas was occurred when the pre-service mathematics teachers interested in students' reasoning. The pre-service mathematics teachers aimed to reveal underlying reasons behind students' ideas. Similarly, Wick and Janes (2006) also stated that the thoughts and ideas of the students were able to reveal with using questioning. Additionally, the teacher's use of questioning throughout the presentations helped to include all of the students in focusing on the different ways to solve the problems (Suurtamm & Vézina, 2010). By asking students' how to reach their ideas, the pre-service mathematics teachers also informed about their students' obstacles, misconceptions and the mistakes related to limit concept. Thus, chance to correct mistakes or deficiencies of the students' ideas would arise.

To correct mistakes, the pre-service mathematics teachers preferred students to find their mistakes themselves, or to discuss mistakes with the others. Rowland (2008) stated that on one occasion when a student gave a wrong answer, the teacher asked another student to give the correct answer instead of trying to establish why the first student made the mistake. The pre-service mathematics teachers sometimes immediately corrected their students' mistakes. Ding and Li (2006) state that some teachers did not care about or recognize students' mistakes, and teachers who did dealt with mistakes in different ways (Ding, 2007). Sometimes the further questions were asked to the students that lead them to correct their own mistakes. By giving feedback to students' mistakes, misconceptions and obstacles etc., the pre-service teachers could help them with their problems and enabled them to discover their mistakes. Suurtamm and Vézina (2010) also paid attention to correcting mistakes and handled student mistakes as opportunities for learning.

Ignoring students' ideas focused on the situations in which students' ideas, answers and comments were not paid attention and ignored. In this study, because of ignoring the participants could not use opportunities. When a learner produced an unexpected contribution, teachers usually did not entertain that response, but continued to look for a response that would be consistent with their thoughts (Mhlolo & Schäfer, 2012). Empson and Jacobs (2008) maintained that by listening to students' strategies and explanations during problem solving, students' mathematical understanding would improve, and the teacher's mathematical knowledge would also increase. In a similar vein, Rowland et al. (2009) stated that the teachers' paying no attention to the students' thoughts may stem from their beliefs that the students could learn in this way, too.

7. Recommendations for practice

The following suggestions were presented on the basis of the findings:

- The importance of paying attention to students' ideas might be handled in teacher education program. For this reason, the sub-codes of responding to students' ideas may be used as a frame.
- The pre-service mathematics teachers could be informed about how to prepare a detailed lesson plan by considering students' possible ideas. After preparing lesson plans, they could teach lessons in real classroom environment or with microteaching.
- When the pre-service mathematics teachers encounter an unexpected problem they should deal with it in the lesson and not ignore it. If they do not overcome the difficulty at the time, they should ask a homework about it. They also determine` the solution themselves and deal with it in the next lesson.
- The pre-service mathematics teachers could be informed that they can come across unexpected events in their lessons like dealing with points in a subject which could be difficult or easy for the students to comprehend, misconceptions or questions posed by the students. It is suggested that this study which was carried out to examine the lessons of the pre-service mathematics teachers in the context of responding to students' ideas could also be carried out for the mathematics teachers.

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