



COMPARISON OF THE CLASSICAL SOLUTION TO SUBTRACTION EXERCISES AND THEIR SOLUTION WITH REPRESENTATIONS

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Abstract:

294 pupils aged 8-9 years were given subtraction problems. Initially the pupils managed to solve the exercises using the usual algorithm ($a-b=c$). Simultaneously they made a representation of their solutions using 4 shapes which had been pre-agreed by the pupils and their teacher. Not only were the results unsatisfactorily worked out, but they were lower than the (also) unsatisfactory solutions given in the students' efforts to solve the problems in the classical way. A teaching configuration was then prepared. After this an overall improvement was discerned in the majority of pupils, in subtraction problems.

Keywords: primary school, representations, subtraction exercises, using the usual algorithm

1. Introduction

From the moment children start going to school they come into contact with a host of representations in mathematics, mainly pictures. Primary school teachers, for their part, try for the understanding of abstract concepts with the use of a variety of representations (books, teacher training, pictures, different kinds of software).

There are important considerations to be made by primary school teachers as to how they can use graphic representations to improve the way they teach their pupils. In this work we examine the difficulties that pupils of the 4th class of Primary School encounter in the solving of exercises in Mathematics that are to do with subtraction. These pupils had already been taught addition and subtraction in the earlier classes of Primary School. There was a simultaneous comparison between the classical way of solving problems and in the solving of exercises using representations of the solution with pre-

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agreed shapes. Then, based on these conclusions, improved techniques in the pupils' way of solving problems were proposed, which were nothing more than the spacing of the exercises that the pupils have to do into specific sections on their exercise sheet, following the usual model of solving subtraction exercises ($a - b = c$).

So, the aim of this work is to investigate whether the use of graphic representations improves pupils' ability to solve problems of addition or subtraction when compared to the usual form of solving such problems where the students simply work out the subtraction sum in the usual way. By 'the usual way' we mean the form $a - b = c$, where the subtrahend b is subtracted from the minuend a to give, finally, the difference c (result). In the second phase we examine the pupil's ability to solve subtraction problems is improved if again the format $a - b = c$ is followed where each letter (a , b , c) is represented by a space where the sums given in the problem are written (as explained below).

In the present work we do not examine the pupils' ability to correctly execute the algorithms of addition or subtraction.

1.1. Research Questions

Can the pupils' achievement in subtraction problems be improved, if we use prearranged schematic representations instead of the classical way of solving subtraction problems?

Is there a difference between pupils' ability to solve the same problems in the classical way and their ability to do so through representations of the solutions?

Are the pupils better able to discern the concepts and also the relationships which exist between these concepts in the problem when they are presented in the classical way and with schematic representations, if the solutions to the exercises appear in the form ($a - b = c$), rather than in any other form?

1.2. Hypotheses

Pupils have greater facility in solving problems of addition and subtraction in the classical way or representing the solutions if they follow the form ($a - b = c$). The representation of the solution to a problem (addition or subtraction) can create greater problems and show lower achievement than does the solution of the exercises in the classical way.

2. Theoretical Approaches

Regarding the theory of the creation of representations there is a series of studies (Elia, Gagatsis & Gras, 2006 Gagatsis et al., 2002 Gagatsis & Christou, 2002 Von Glaserfeld, 1987). There are also significant studies referring to the translation of representations (Elia, Gagatsis & Demetriou, 2007 Gagatsis & Christou, 2002). When we say "*translation of representations*" we mean a psychological process which takes place when we transfer from one system of representations to another, such as for example, from an algebraic

equation to a graphic representation (Gagatsis & Siamari 2003). For this transfer from one system to another, as Lesh maintains, (Lesh, 1979) at least two forms of representation are necessary: the source – the initial representation - must be pictured from the ‘point of view’ of the second.

There are very many studies which refer to internal and external representations (Elia, Gagatsis, & Demetriou 2007 Lesh, Post & Behr, 1987). Internal representations are defined as the total of intellectual pictures, thoughts or expressions which allow the individual to connect data, to separate main facts from subordinate facts, to connect knowledge from different topics and times, to find possibilities and alternatives, and to analyse and connect steps in logic. External representations comprise all organizations of exterior symbols (shapes, symbols, diagrams), and have the objective of representing externally a mathematical reality. In this must also be included the signals an individual uses to express the concept or a situation, such as oral or written words in physics or in artificial language, symbols drawings and pictures (Avgerinos & Marinos, 2009).

The representations that we mention as being used by the pupils take various forms (symbolic – numeric – lexical – pictorial).

Translation between different representations of the same article can be considered a necessary prerequisite for the solution of mathematical problems (Duval, 2005). Moreover, ability to connect different forms of representation to a common concept is of great assistance to the pupils’ comprehension (Gagatsis & Shiakalli, 2004 Griffin & Case, 1997). It also helps them to be able to describe the entirety of a mathematical concept or structure. By using multiple representations which refer to the same concept the pupil can conceive the common attributes of different representations and can evaluate them since the recognises structural relationships between different situations by using external characteristics and in this way can better approach mathematical knowledge (Greer & Harel, 1998).

Duval had much earlier maintained that every semiotic field has different possibilities. A shape for example cannot be said to have the same possibilities of representation that language has. Thus, with shapes we can represent the entirety of the relationships that exist between the individual elements which make up an object or a situation. From another point of view shapes and a portion of representations in general represent only situations, formations or results of actions, without being able to represent actions or transformations where a field which has the attributes of the language of physics or algebra is required (Duval, 1987).

Various studies have shown the positive results on learning in the educational process that representations have (Elia, Gagatsis, & Gras, 2006 Horton et al., 1993). The effectiveness of representations of concepts has been compared with various other learning techniques. For example, pupils who used a representation of a concept as a strategic learning tool have better results in solving the exercises which were set compared to other pupils who used the classical way of solving exercises (Robinson and Kiewra, 1995 Panaoura, 2009 Chularut and DeBacker, 2004), while in parallel we have an improvement in their comprehension (Davis, 1990).

According to Gagatsis and his colleagues (Gagatsis et al., 2001) it is good for pupils to try different ways of solving exercises. Specifically, they argue that if exercises are given to the pupils and they try to solve them in different ways, this can help them to handle the problem from a different optical angle and to discuss suitable representations which will help them to deal effectively with the problem. Indeed, Gagatsis adds that there are situations in which translation from one code of representation to another is automatic (Gagatsis et al., 1999). The significant work which representations offer can become even greater if, in agreement with Stenning and Oberlander, (Stenning and Oberlander, 1995) schematic representations are accompanied by full explanations and linear connections so as to make things easier for the pupil to represent his thinking during the solving process (Panaoura, 2009 Gagatsis, Agathangelou, & Papakosta, 2010), Marinos and Avgerinos, 2012a).

One of the positive elements of the acquisition of knowledge through the aid of representations is that the pupils investigate their own cognitive structures. And so, this paradox is observed: While the teacher uses various pictures or diagrams to make his/her teaching more comprehensible, this can also create more difficulties for the pupils.

Apart from the positive elements offered by a representation, however, the use of representations can also have negative effects. That is to say representations, instead of making things easier for pupils, can create greater problems of comprehension (Colin, Chauvet and Viennot, 2002; Bishop, 1989). These difficulties can be the focus of attention to details of the representation which are not relevant to what is being asked in the exercise (Presmeg, 1986). It can even be that the picture is not equivalent to the problem, and often elements can be connected with several meanings, or the spatial provision of the picture can be unsuitable (Colin, Chauvet and Viennot, 2002). There is also the situation where the same concept could simultaneously have the same or an oppositional structure or have different directionality of dimension. Many times, too, people do not understand that the different representations show two different ways in which the problem can be solved, that is to say that they are tools which help them practically in solving the exercises (Marinos, Avgerinos, 2012b).

A further limitation to their attempts to solve the problems with representations can be explained by the help of the "Theory of Cognitive Loading" (Sweller, Merriënboer, & Paas, 1998, Van Merriënboer, & Sweller, 2005 Sweller, 1988). According to this theory a pupil has more difficulty if, on the one hand, he is influenced by and has learned to solve the exercises in the classical way but is trying to solve the exercise (representation of the solution) in a new way which is completely foreign to him.

So, teaching of different ways of solving the exercises should precede the actual task of solving, taking into account the possibilities and limitations which exist. For this reason, it would be good for students to learn to move step by step from one representation to another, a fact which must be faced with special care by teachers according to Dufour-Janvier (Dufour and Janvier, 1987).

An important study that refers to the actions and processes of addition and subtraction with pupils' schematic representations, is the one carried out by Selter (Selter,

2001). It is worth noting that Vergnaud and Durand began research into problems of addition and subtraction in about 1976. Since then, there has been much discussion to do with the complexity and the wide variety of exercises as well as the composition of the cognitive functions that at first glance appear easy (Vergnaud and Durand, 1976).

Rina Hershkowitz and her colleagues point out that the act of subtraction requires theoretical thinking, while Davydov formulated the opinion that subtraction can also contain elements of empirical thinking. Thus, a subtraction process can lead from the initial unrefined abstract entities to a new structure (Hershkowitz et al., 2001).

The usual type of subtraction problems can be said to belong to one of the following categories:

- the subtraction may appear as a remainder. In this case we give the pupils a quantity where a part is subtracted and they are asked for the remainder.
- the subtraction may appear as a comparison. Here we have two amounts which the pupils subtract finding the difference between them.
- the subtraction appears as a balance. In this case the problem contains two quantities and the pupil is asked to reduce the greater quantity so that it counterbalances the smaller
- the subtraction is an omitted addition. Here the pupil develops the spoken deductive logic and then makes some analysis and comparison of logical suggestions and relationships (Philippou-Christou, 2002).

3. Data Collection

As well as the problems that were given to the students, in order to check their knowledge of mathematical concepts they were also given a test to check their knowledge of subtraction sums before we began the process and again at the end, once more using Semi-structured Interviews. The interviews were semi-structured to allow the teacher to record from each student the feelings generated by an approach to mathematical concepts with the help of graphic representations and were also recorded on Video-Dvd. Following some indications of different authors about the use of videos both from a general point of view and other more specific points coming from educational research in mathematics, we videotaped the teacher teaching a whole unit of his own design. Videotaped classroom data were collected in the teacher's classroom by placing a camera at the back of the room. During whole-class discussions, the camera captured the activity of the teacher and the students who showed their work on the blackboard located in the front of the classroom. During small-group work, the camera followed the teachers and captured their interaction with the different groups.

The evaluation took place using 10 points on a scale from 1 to 10. Students who appeared to be unable to solve the exercises algebraically or by sketching their solutions were given grades below 4. Students who didn't even try to solve them were marked as 1, while those who were excellent at solving them were assessed as 10. To discover if the concepts of mathematics with addition and subtraction had been acquired, we compared

the results before with a pre-test and then after the teaching which presented the concepts of subtraction with shapes.

4. Sample

The sample of the quantitative part of the study consisted of 294 fourth graders in Primary School. The study included females ($n = 95$) and males ($n = 199$) between the ages of 8–9 years old. The research took place in 4 schools in a large island city with the participation of 9 classes.

In total, eight teachers (seven male teachers and one female teacher) with experience ranging between 11 and 25 years participated in the study. The instructions and phases as described below were common to all the classes and combined a narrow step-by-step procedure with a high density of knowledge transfer. One researcher participated, who gave out the exercises and necessary clarifications to all classes in order to ensure that the instructions that were given were the same for all.

As we know, the pupils had been taught subtraction sums in ordinal numbers since the first and second years of primary school.

In the mathematics books that are taught in primary school, but also speaking generally, the host of graphic representations serves 4 functions in the solving of mathematical problems: decorative, auxiliary-representational, auxiliary-organisational and informational (Gagatsis & Elia, 2003)

The pupils were given worksheets, and class observation was by video recording of the lessons.

The participating teacher / researcher also made parallel use of notes.

A series of symbols were presented to the students on the chalkboard which would represent quantities that they possessed or lost, and also sums such as addition or subtraction which would symbolize the results etc. The students agreed to use the symbols described in Figure 1. But after this agreement the teacher asked each of the students separately what the shapes in Figure 1 represented and how each student would use them in various problems.

We must clarify that the reason these students were asked to make this choice is that they each arrived at the school with a very informal or merely intuitive knowledge of mathematics, modified or strengthened by photographs, schematic diagrams etc. that they had come across in their books or through various electronic media (for example on computers). This knowledge could be used as a basis for a large part of the scientific mathematics in the primary school curriculum, provided that the teacher can use it in a suitable way, connecting informal and formal knowledge (Carpenter, Fennema, Franke, 1996).

We tried to make it so that the formulation of the exercises that were given to the pupils could be created in their minds from a representation of the solution. Thus, the solution of the exercise in the customary way could more easily be connected with the solution which included graphic representations constructed by the pupils.

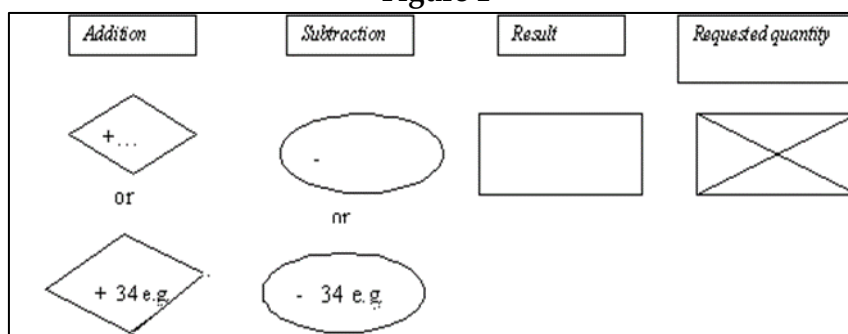
Three teaching hours in all were dedicated to each class, divided into three phases.

Phase 1: in the first part the pupils tried to solve the exercises they had been given in two ways – either the classical way or with schematic representations. We evaluated their ability to solve exercises in the classical way and with representations of the solution with the help of diagrams which the pupils constructed. For the second part of this phase one and a half periods (60 classroom minutes) were dedicated to the students' making representations of the exercises and also using classical algorithms.

Particularly for Phase 1 Part 2 for solving the problems 4 shapes were used which had been pre-agreed with the pupils.

The rhombus with the symbol for addition and the oval with the symbol for subtraction of known or unknown quantities which the pupil should add or subtract, in order to get a result, probably a known.

Figure 1



That is to say, when the pupils want to add and don't know the quantity, they can use the symbol $+...$ inside the rhombus, then when they want to add a specific quantity e.g. the number 34, they again use the same symbol placing the added number as well as the symbol inside the rhombus.

The following recommendations were followed:

- to place the concepts in a structure which, in their opinion, better represents the solution to the problem.
- to use straight lines with simple or double arrows to connect the terms and concepts which are related.
- to use words or phrases as labels along the lines to state the relationship between two connected terms.

The pupils were given the following problems and asked to solve them in the classical way and then to solve the exercises with the help of schematic representations.

Phase 2: in this phase the teaching configuration included the preparation of a suitable provision of shapes in columns. These were simply the form $(a - b = c)$. Each space represented a letter. For example, in the space A were placed the amounts which had referred to in the exercise, in the space B they placed the quantities to be subtracted while in space C they placed the result. Thus, if they already had an amount, or they gained an amount the students placed it in space A, if they took away an amount then they used

space B etc. The action between the different spaces like a and b or a and c would be the subtractions. $(a - c = b)$ or $(a - b = c)$

The student could also do sums of addition using the spaces, (a or b) when, for example the exercise said: «had» or «gained» (problem 2), then the student placed the amounts which had been gained in space a.

So, in exercise 3 the number of marbles the student won were placed in space A, while in space C (result) the number 18 would be placed as the number of marbles he would finally have.

In a similar way if a student lost an amount in the beginning and later lost a further amount, these amounts would be added and placed in space B (problem 4).

If a student began to play with a certain number of marbles and at the end was left with another number of marbles, he placed the initial number in space A and the number he had at the end in space C, again making the subtraction in the form $a - c = b$ (problem 5).

One period (about 40 classroom minutes) was dedicated to the teaching configuration on the same day as the earlier evaluations and using discussion as a means to ascertain whether or not there had been a change in the pupils' comprehension of the exercises.

Phase 3: in this phase there was an evaluation. The duration about 40 classroom minutes to this after one week in order to check the assimilation of the lesson which had been taught, using similar problems.

5. The Problems

The pupils solved each exercise in the classical way and then they solved them with the help of conceptual maps.

- 1) Constantina's grandmother gave her 476 Euros on her Name Day. Constantina spent 177 Euros on a toy. How much money did she have left?
- 2) Maria had 47 cards. On Monday she won 16 cards in the game that she was playing with Dimitra, but on Tuesday she lost 17 cards. How many cards did she have left altogether?
- 3) George had 18 marbles, and Dimitris had 21 marbles. They played two games of marbles. In the first game Dimitris won 3 marbles. How many marbles altogether must Dimitris lose in the second game for George to catch up with him?
- 4) Michalis loses 13 marbles in the first game and in the second game he loses another 19 marbles. How many marbles did he lose all together?
- 5) Nikos has 5 marbles and plays two games. In the first game he wins 5 marbles. At the end he finds that he has 3 marbles altogether. How many marbles did he lose in the second game?

5.1 Note

The students solved the exercises in two different ways. Initially they used the way that was already known to them then they drew the solution to the problem using the shapes that we had agreed.

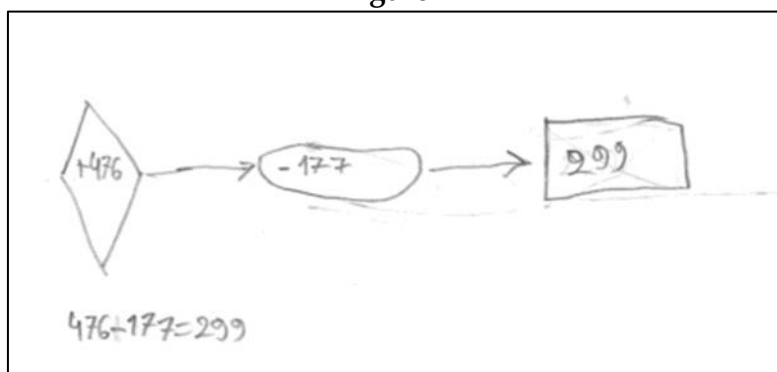
In order to process the elements after the research was completed, we used a system of coding for the representations that the participating pupils had prepared. A similar thing was done also by Rafferty and Fleschner (Rafferty, Fleschner, 1993). Thus, throughout the evaluation whatever concerned the solving of the exercises with schematic representations the following was taken into consideration:

- a) The correctness of the concepts as represented in the schematic representations prepared by the pupils.
- b) The number of wrong concepts (shapes) that the pupils used (regardless of the previous paragraph).
- c) The number of correct connections.
- d) The relationship of classical solutions to the representations made by the pupils. The solving of the exercises in the classical way had also to be connected to the design that the pupils made. In the final question the pupils write a number from 1 to 4 on a scale.

6. Results

1) To the problem of Constantina who was given 476 Euros by her grandmother on her Name Day then spent 177 Euros buying a toy, so how much money was she left with? The subtraction here appears as a remainder. There is practically no difference between the results from solving this in the classical way and the results from solving it with a representation.

Figure 2



It can be seen (Table 1) that the pupils very easily found the solution to these particular exercises making calculations $M=7.89$ $SD=1.19$ There is a very significant difference since $t=-4.24$ $p=0.000$ compared the pupils who solved them using schematic representations $M=6.26$ $SD=1.326$ (Figure 2).

Table 1: Evaluation of students in the classical solution of exercises and in the solution of the same exercises using graphic representations

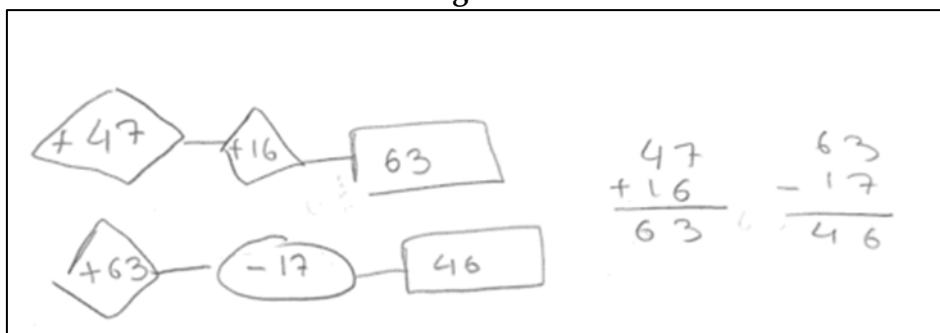
Exercise number	Solution of exercises in the classical way		Solution of exercises with graphic representations		t	p
	M	Sd	M	Sd		
1	7.8947	1.19697	6.2632	1.32674	-4.249	0.00
2	6.1579	1.06787	4.8947	2.42429	2.317	0.33
3	4.4211	1.88108	2.4211	1.30089	13.077	0.00
4	8.9474	1.22355	6.9474	1.84010	6.164	0.00
5	8.1579	2.11511	5.9474	2.43752	-3.553	0.004

Note: The assessment was carried out on a ten point scale.

2) As regards the second exercise (Figure 3) we observe that there is no significant statistical difference between the classical way of solving the problem and solving it with a representation of its solution, since we have $t=2.317$ $p=0.33 > 0.05$.

The average of the pupils who solved the exercise in the classical way was $M=6.157$ $SD=1.067$ in relation to those pupils who solved the exercises by representing their solutions $M=4.87$ $Sd=2.42$.

Figure 3

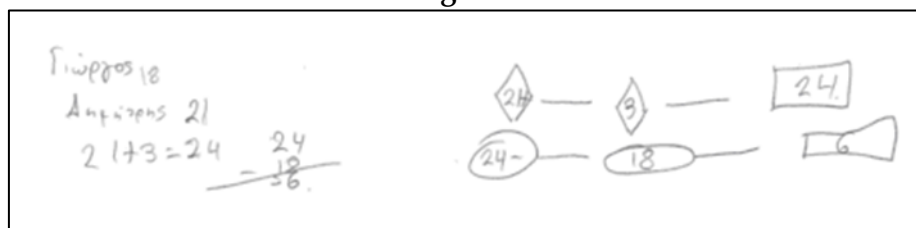


3) To the problem: George had 18 marbles, and Dimitris had 21 marbles. They played two games of marbles. In the first game Dimitris won 3 marbles. How many marbles altogether must Dimitris lose in the second game for George to catch up with him?

The subtraction here appears as a comparison (two quantities that the pupils subtract finding the difference between them).

It can be seen that in their attempt to solve the exercise in two ways the students generally had significant difficulty. Here, however, the most significant difficulty was in their attempt at solving the exercise with schematic representations $M=2.42$ $SD=1.30$ (table 1) compared to the pupils who tried to solve them in the classical way ($M=4.42$ $SD=1.88$). Apart from the low results for both exercise solving forms it can be seen that here, too, there is a very significant statistical difference between the two forms since $t=13.077$ $p=0.000 > p=0.001$ (large statistical difference) (Figure 4).

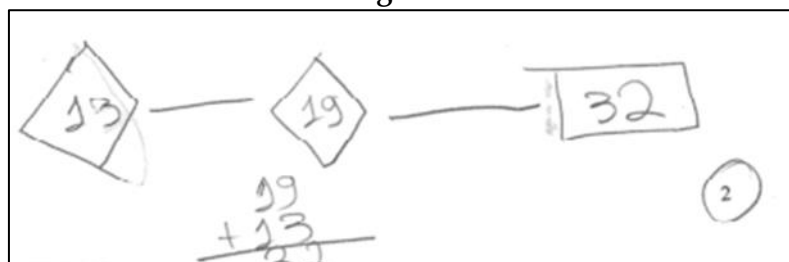
Figure 4



4) To the problem: Michalis loses 13 marbles in the first game and in the second game he loses another 19 marbles. How many marbles did he lose all together?

In solving this exercise, the pupils did not have difficulty either with the classical method or with representing the solution. It is noted that those who managed to solve the exercise in the classical way could also do so by representing its solution. The results of the evaluation were $M=8.94$ $SD=1.22$ compared to the solving with schematic representations $M=6.94$ $SD=1.840$, the pupils solving the exercise by calculations having much less difficulty than those using the schematic method since, noting a statistically significant difference since $t=6.164$, $p=0.000 > 0.001$ (large statistical difference).

Figure 5



5) To the problem: Nikos has 5 marbles and plays two games. In the first game he wins 5 marbles. At the end he finds that he has 3 marbles altogether. How many marbles did he lose in the second game?

Here the subtraction appears as an omitted addition. It is seen that here too the pupils again have less difficulty carrying out their calculations ($M=8.157$ $SD=2.11$) in relation to solving with graphic representations $M=5.94$ $SD=2.43$, noting a significant statistical difference $t=-3.55$ $p=0.04$.

From all the above it appears that pupils have more difficulty solving the exercises graphically than they do in solving them algebraically.

6.2 Teaching Configuration Phase

The researcher asked the pupils to divide their worksheet into four columns and to name each column as below (figure 6), following, as extensively mentioned above, the form: $a-b=c$, that is to say we asked the pupils to place the concepts of the problem in the appropriate column to match the headings which we gave to each column. It has made clear to the pupils that once again they could use straight lines to join to terms and concepts in order to arrive at a solution to the problems, as well as placing words or

phrases along the lines to explain the relationship between the connected terms. It should also be noted finally that the problems that were used in this section of the research was the same as those which were used in the earlier phases.

Figure 6

I have	I win/am given	I lose	Result

6.3 The role of the researcher/ teacher in this phase

Below we mention some of the questions that the teacher asked the pupils, in order to lead the students to cognitive conflict and finally to select the right answer.

- Why did you place the shapes in this way this time?
- What was it that made you make these changes?
- Mention the reasons which lead you to specific solutions. For example:
- While the problem tells you that we have lost some quantities, why have you done an addition?
- In the first phase you used the shape which shows the quantity that you lose, why in this phase have you placed this shape in the column that refers to quantities that you have won?

Results after the teaching formulation based on the form $a - b = c$. It can be seen that the above model separating the pupil's work sheet into sections following the form $a - b = c$ had a positive result for the subsequent solving of similar exercises. Thus, it can be seen that when the subtraction appears as a remainder there is a great statistically significant difference since $t = -4.68$, $p = 0.000 < 0.001$ (a very large statistical difference).

A similarly statistically significant difference exists when the subtraction appears as a comparison. It may be seen then, in similarity with the form of exercises 2 and 3 there is a statistically significant difference in ability to solve the exercises $t = 1.34$ & $t = 2.316$ and $p = 0.004 < 0.05$ & $p = 0.00 < 0.001$ respectively. There is a similar great improvement in the pupils (Table 2) when the subtraction appears as an omitted addition (as in the 4th and 5th exercises). In this case here is a great statistical difference since $p = 0.000 < 0.001$ (a very large statistical difference).

Table 2: Evaluation of the solution of similar subtraction problems with graphic representations before and after the teaching formation based on the form $a - b = c$

Form of Subtraction	Solution of exercises using graphic representations before the teaching formation		Solution of similar exercises with graphic representations		t	p
	M	Sd	M	Sd		
The subtraction appears as a remainder (as in the 1 st previous exercise)	6.2632	1.32674	8.0000	1.15470	-4.860	0.000
The subtraction appears as a comparison (as in Exercise 2 and 3)	4.8947	2.42429	6.6316	1.34208	-3.250	0.004
	2.4211	.50726	5.6316	2.36198	-5.683	0.000
The subtraction appears as an omitted addition (as in exercise 4 and 5)	6.9474	1.84010	8.7895	1.47494	-5.351	0.000
	5.9474	2.43752	7.6316	1.92095	-4.491	0.000

Note: The assessment was carried out on a ten point scale.

A significant improvement is observed also in the solution of similar exercises by calculation.

The results with Anova show the combinations for three categories of data, i.e. solving the exercises in a schematic way, improvements, and solving the exercises with calculations.

Thus, it can be seen that when the subtraction appears as a remainder there is no statistically significant improvement from the initial solutions that the pupils found using calculations $F(2.72) p=0.072 > 0.05$, nor from the schematic solution the pupils found before the teaching formation, since $F(0.60) p=0.669 > p=0.05$ (so that we have a significant statistical difference). Finally, there is no statistically significant difference in the solutions with calculations for similar exercises since $F(1.87) p=0.171 > p=0.05$ so that we have a significant statistical difference). The pupils could, as we mention above, record very good results in solving this kind of exercise (where the subtraction appears as a remainder).

A statistically significant difference between the classical way of solving the exercises and the solving of them by schematic representation when the subtraction appears as a comparison (as in the 2nd and 3rd exercises) can be noticed from the initial solution that the pupils carried out using calculations $F(6.44) p=0.004 < 0.05$. There is also a statistically significant difference with the evaluations of the schematic solution of the exercises before the teaching formation since $F(8.48) p=0.001$. Finally, there is a statistically significant difference also with the results of the evaluation of solution $F(69.44) p=0.000$

A statistically significant difference is not apparent between the solutions of the exercises when the subtraction appears as an omitted addition in the classical way before the formation of teaching since $(F=2.44) p=0.118$. There is however a statistically

significant difference between the schematic solution of the exercises before the teaching formation $F(12.69)$, $p=0.000$ and after $F(23.257)$ $p=0.00$

7. Conclusions

We divide the conclusions into two subsections:

7a. Initially (in the 1st section) we will give some explanations about the difficulties which the pupils met in solving problems with representations of their solution. In this section will mention some conclusions which are related to the effect of the different places for the unknown as well as for the subtrahend and minuend in the solution of subtraction problems with the help of distributed worksheets.

7b. Next (in the 2nd subsection) we note the observations that have to do with the placement of the schematic representations in specific places on the worksheet.

7a. Difficulties encountered by the students in solving problems with representations

1st subsection of the conclusions: it was observed that the pupils had few difficulties in using the representation effectively when the subtraction appeared as a remainder. Thus, they managed to solve the problem also in the classical way. We believe that this success is due to the fact that the mathematical structure of the problem is easy and so the creation of both a graphic representation of the solution to the exercise and the solution of the exercise in the classical way are feasible. These problems, too, which have an unknown as the final quantity can easily be modeled either by the teaching plan or in school textbooks as well as in the teacher's teaching plan, or even from the results of the pupil's own thinking, and they present semantic agreement.

The second exercise also presented no difficulty in the classical way of solving it (i.e. the pupils could create the necessary sums and then solve them), nor in the construction of schematic representations of the solution. We consider success in solving the exercise to be due to the fact that this, like the previous problem, is a subtraction problem with an unknown as the final solution. Again, for most of the pupils, the initial number of cards that Maria had did not cause any than any difficulty in solving the problem.

In the 3rd and 5th exercises with George and Nikos playing marbles, it was observed that the pupils were not in a position to solve these problems. This was because in these two problems they have to make a transfer of terms, which requires the simultaneous use of two actions. Parallel to this, the fact that the unknowns are in different positions from the usual (i.e. they are neither in the familiar places of the beginning nor at the end of the subtraction, and so the pupils have to transfer the terms of the subtraction or the addition), caused them difficulty. These problems also presented great difficulty since there had to be a hierarchy in the intellectual activity which took place in the pupils' minds. These exercises required intellectual processes which were beyond pupils of 7-8 years old, and again, there were difficulties because of semantic non-agreement between the written and verbal representations of the solution.

The findings of the 1st Phase of the research are indicative, where it is shown that a significant number of the pupils who correctly used the classical solution of the exercise were unable to create a representation. The findings of the research also show that the creation of a representation is difficult for the pupils. In these exercises it is worth noting that the theories of Daour-Javier 1987 were verified. In accordance with these theories, the use of schematic representations of the solution as a tool helped the pupils to find the solution to the exercises, when they saw it as a problematic situation. These results (as is shown in the 1st Phase) are not in accord with the hypothesis of Waller (Waller, 1981 in Vekiri, 2002), nor with the references of Paivio which have to do with the theories of binary codification (Paivio, 1990). Nor are they in accord with the findings of Mayer, Mayer & Gallini (Mayer Mayer & Gallini, 1990) where they maintain that representations that have to do with pictures and shapes and connecting pictures and shapes with the written text can help the comprehension of the pupils and lead them to the solution of the problems they are given.

Thus, we observe that in our own case, and after the first phase of the research, the theories of Colin and his colleagues, and Bishop were further verified (Colin et al, 2002 & Bishop, 1989), according to which the use of the aforementioned representations bring about negative results, causing greater problems for the pupils. These obstacles are probably due to the fact that the present schematic representations, despite the pupils' acceptance of them, are not often met with in the textbooks in the framework of the solving of subtraction problems. Thus, the use of schematic representations does not constitute a familiar situation for the pupils.

7b. Placement of the unknown in a particular place

In the second phase of the research – The second intervention of the researcher/ teacher, the difficulties which emerged from the first phase, the placing of the representation in a specific place, were dealt with ($a-b=c$ or $a+b=c$). This placement is important not only for educational reasons, but also because it offers the required theoretical background for an argued critique of the “facilities” which the representations provide.

Thus, by handing out worksheets to the pupils, the determination of the unknown in a constant place makes it easier for the pupils to solve the problems. In our own case a “facility” is considered to be a positive contributory factor to the construction of a meaning or α concept if, through this, comes the possibility of printing a problem in the fixed form which a representation can take. That is to say, the dividing of the pupils' worksheets into set spaces. For example, when the exercise mentions that there is already a quantity, the pupil can place it in the space marked “I have” along with the symbol which was used in the earlier phase, and which was agreed with the teacher. Similarly, if the exercise gives a result and asks if the pupil has gained or lost something he should place the equivalent sum with its symbol in the “results” space and should try to see if he has gained or lost. That is say the splitting of the worksheets into sections works to enhance the optical perception and also to reorganize the pupil's thinking (Pea, 1985; Dorfler; 1993 Koleza). The pupils are helped to solve the problems in which they have

had difficulty, through a combination of discussion, the placement of the unknown in different places for different representations, and by passing from one representation to another for the same mathematical problem. This strengthens the importance of the model, which is supported by an interaction of different ways of representation through the different placements of the unknown.

The use of such a procedure encourages the semantic approach which is supported by structural elements, and the relationships of the problems which are, themselves, combined with a representational approach, using different the various representations and then changing from one to another. It helps the pupils significantly at this age if they are allowed to accompany the shapes with lines, labels, explanations, connections, etc.

In the lessons which followed the pupils did exercises on the 4 sums using partition of their worksheets into sections, where each section connects with a specific arithmetical sum or situation.

In the re-evaluation which was carried out after one month the pupils were examined in similar problems to those used in the first evaluation.

In the evaluation of their solutions, it is observed that the pupils as a whole had improved in both the classical way of solving the exercises and also in the way of creating a representation of their solutions.

Specifically, we evaluated that the pupils were assessed at $a=9$ for problems with a remainder, while solving the exercises using representation of their solutions, they were assessed at $a=8.8$. The cross-correlation index of the two forms was $r=-0.810$.

Where the subtraction appeared as a comparison, we observed that the classical way of solution was $a=6.71$ while representation of the solutions was $a=7.29$. The cross-correlation index of the two forms was $r=0.4$

Where the subtraction appeared as an omitted addition a satisfactory solution of the exercises at $a=7$ while with the classical way of solving the exercises was $a=5.86$ in the cross-correlation index of the two forms which was $r=0.723$.

To sum up apart from the fact that we had agreed with our pupils the ways of representing the solutions, we did not observe any essential change in the solutions of the exercises from the classical way. On the contrary, we observed greater difficulty in certain problems. Something of this kind is in accordance with many other studies which refer to the theoretical framework of the task. These studies have shown how complex such a task is, since the pupils have difficulty in the application and comprehension of shapes, and also in handling the different shapes easily so that they represent the solutions (e.g., Friel, Curcio, Bright, 2001). This happens in nearly all the problems where the pupils were asked for representations, if we exclude the 1st, 2nd, and 4th problems which did not cause them any difficulty for reasons which we have already mentioned above. These problems where the solution is encountered as a "remainder," seem to be considered much simpler and the pupils can find the solution easily using either method. Thus, in the first part of the research it seems, for the reasons stated above, that representations constructed by the pupils are not a panacea for the solution of all problems of addition and subtraction.

In the second part, after the teaching configuration, the technique that was asked was the division of the page into four lengthwise sections. Shapes were placed in these 4 sections and explanations were given that were clearly better than those which had preceded this teaching. This was because the pupils have represented the subtraction sum in their minds as $a-b=c$. Then interpretation of the representations created an activity (Roth & Bowen, 2001) where the pupils could understand the relationship between representations and the area that they represented. Essentially, by configuring our teaching in this way, we tried to help the pupils to codify the data. We also taught them how to observe things better, to evaluate the words in the statement of the problem, and how to demarcate the concepts on the work sheet they were given so as to develop most effectively the information that was asked for. That is to say, the opportunity was given to the teachers to show the pupils how to codify the information they are given by converting the problematic situation into a solution. We further observed that the pupils should understand how to construct a suitable representation, and that this understanding could take place through the teaching of a technique. Then we observed (during the teaching configuration in the second part of the research) that after teaching of such a technique the pupils succeeded in drawing the representations of the solutions to the exercises. We further believe that the choice of suitable representation is more difficult for pupils whose teachers have not worked on or used elements of representation in their teaching plans. This also has to do with the fact that the teachers themselves do not have in-depth knowledge of the objectives they are trying to resolve (Chi, H., Feltovich, J. & Glaser, R., 1981).

The benefits from a representation cannot be conceived immediately, since initially we observe that the pupils, instead of finding things easier, come face to face with more complex problems than the one they are trying to solve. That is to say, the pupil does not only have to solve the problem in the classical way, but also to be able to use schematic representations which will explain the solution to the problem.

Conflict of Interest Statement

The author declares no conflicts of interests.

About the Author

Andreas Marinos, son of Christos, was born in Rhodes-Greece in 1966. He has a PhD in the Didactics of Mathematics and Evaluation Post-Ph.D in the Administrator with mathematics tools in Education with and post graduate qualification in School Administration. Has studied Electrical Engineering teacher and a graduate of the School of Education of the University of the Aegean. He has over 100 scientific published, all with the critique system. He has presented his own work at 16 International conferences. In 2000- 2001 and 2003-2007 he taught in the School of Education of the University of the Aegean (Gov. Gazette) From 2012-2017 with the Higher School of Pedagogical and Technological Teaching Placement Contract he taught the following lessons: Organisation, Management and Sociology of Education, Educational Assessment, the

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Andreas Marinos
COMPARISON OF THE CLASSICAL SOLUTION TO SUBTRACTION
EXERCISES AND THEIR SOLUTION WITH REPRESENTATIONS

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