European Journal of Education Studies
ISSN: 2501 - 1111
ISSN-L: 2501 - 1111
Available online at: www.oapub.org/edu

DOI: 10.46827/ejes.v11i1.5171 Volume 11 │ Issue 1 │ 2024

EQUIVALENCE OF PARABOLA'S DEFINITIONS WITHIN MATHEMATICAL MACHINES SIMULATED BY "GEOGEBRA"

Georgios K. Ntontos[1i],
Eugenia Koleza[2]
[1]9th Lyceum of Patras,
62 Australias 26442 Patras,
Greece
orcid.org/0009-0004-7244-8637
[2]University of Patras,
26504 Rio, Achaia,
Greece
orcid.org/0009-0000-8211-1924

Abstract:
This paper aims to highlight two historical parabolographs and a third one that is at the design level and to prove the equivalence of the respective definitions of parabola they contain. Specifically, using the Geogebra math software, each mechanism is attempted to articulate another, so that by moving the cursor of the original mechanism to the common output of the two articulated mechanisms, the same parabola is drawn. This proves the equivalence of definitions of parabolic curve as well as of suitably articulated mathematical machines.

Keywords: parabolograph, mathematical machines, definitions of parabola, Geogebra simulations

1. Introduction

The mathematical instrument was a clearly defined category of scientific instruments, common in academic, artisanal, and commercial contexts, and its development from the 16th to the 18th century was largely independent of other instrument categories, during a period when a "scientific instrument" was unheard of (Bennett, 2011). This article describes two historical parabolographs, specifically those of Bonaventura Cavalieri (1632) and Frans van Schooten (1646), as well as a parabolograph designed monographically by Bergsten within a dynamic geometry software environment, presenting a challenge in mechanical construction.

They are "equivalent mechanisms" consisting of properly articulated rods and carrying out a certain level of pointwise geometric transformation. Each mechanism

---

[i] Correspondence: email gkdodos@yahoo.gr

incorporates a different definition of the parabola and is mechanically equivalent to the others. Their mechanical design doesn't focus on speeds, frictions, stability, and wear of each mechanism, but only on transferring motion from the "input" to the "output" and on the way the rods are articulated. Furthermore, the design is "monolinear" and in a dynamic geometric environment, because in simple mechanisms, with the help of monolinear design, the way of construction, articulation, and functioning (mobility) of the mechanism becomes understandable.

The aim of the article is to confirm the equivalence of the specific definitions of the parabola through the "mechanical emergence" of each mechanism within the structure of the other, both precisely tracing the same parabola. Furthermore, the aim includes the educational use of mathematical machines to highlight mathematical concepts through the design and composition of these mechanisms.

2. Bonaventura Cavalieri mechanism

Bonaventura Francesco Cavalieri (1598 to 1647) was an Italian Jesuat mathematician known for his work on problems in Optics and Motion, as well as for the "Cavalieri's principle," which laid the foundations for integral calculus. In his first book studied mirrors (Cavalieri, 1632). He describes his parabolograph in Chapter XLV, page 187, with the translated title "*How the parabolic curve is described through solid instruments, which consist of rules, being the second method in the plane*". The idea behind the construction of the mechanism originates from Proposition II.14 and Proposition VI.8 of the Elements.

The three right triangles ARL, RLK, and AKR are similar because the following comparisons can be made: Angle RAL=angle LRK = ϕ, as acute angles with perpendicular sides, and angle LRA=angle LKR=90°-ϕ=ω (Figure 1). Therefore, the following proportion arises:

$$\frac{\mathrm{RL}}{\mathrm{LK}} = \frac{\mathrm{AL}}{\mathrm{RL}} \Leftrightarrow \mathrm{RL}^2 = \mathrm{AL} \cdot \mathrm{LK} \Leftrightarrow \frac{\mathrm{RL}^2}{\mathrm{AL}} = \mathrm{LK}$$

From the equation, it is evident that the continuous curve traced by the stylus of point R is a parabola with vertex A, axis of symmetry AM, and "latus rectum" equal to the length of segment LK. The semi-line AK (Figure 3) becomes a diameter passing through the vertex A of each parabolic arc RAQ and bisecting every parallel chord RQ according to Proposition 14. Thus, for every position of R, as the cursor LK moves, the ratio of the square of the distance RL to the corresponding distance AL is always equal to the fixed length of the dragging rod LK. By placing the center of the orthonormal coordinate system at point A of the mechanism plane and considering the vertical axis y'y, which coincides with the line passing through the sliding groove, an attempt will be made to describe the machine algebraically (Figure 1). Furthermore, by defining (x, y) as the coordinates of point R and c as the length of segment LK, the relationship obtained from the geometric reference framework is transformed for the simple Cavalieri's Parabolograph as follows:

$$\frac{RL^2}{AL} = LK \Rightarrow \frac{x^2}{-y} = c \Rightarrow x^2 = -cy \Rightarrow |x| = \sqrt{-cy} \underset{x>0}{\Rightarrow} x = \sqrt{-cy}$$

The point $R(x,y) = R\left(\sqrt{-cy}, y\right)$, and during the operation of this "guiding mechanism," a planar pointwise geometric transformation is performed, given by $L(0,y) \to R\left(\sqrt{-cy}, y\right)$ which is also normal and invertible transformation (Pneumatikos, 1974).

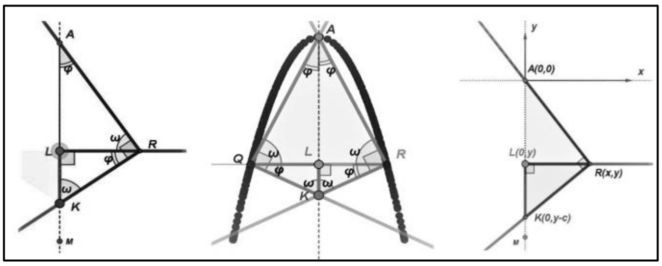

Figure 1: Analysis of simple and complex Cavalieri's parabolograph (https://www.geogebra.org/m/znqdkzsp).

3. Frans van Schooten mechanism

Frans van Schooten (1615-1660) played an important role in disseminating the mathematical methods of Descartes. He achieved this by publishing a Latin translation of "Geometrie" in 1649 with his own comments and explanatory notes, along with related texts from other mathematicians. In 1659, he released a second edition with additional comments and texts. In the notes for the 1649 edition, van Schooten emphasizes the importance of Descartes constructions. Cajori (1893) and Malet (1996) report that van Schooten advised his students, particularly Huygens, to study Cavalieri's theory of "indivisibles."

Van Schooten (1646) correlates the results of the Greek theory on conic sections with his own ideas for drawing curves using motion, as well as constructing corresponding instruments. He explains that the main goal is to describe conic sections in the plane with the aid of motion. According to van Schooten, the precise design of conic sections necessitates incorporating motion. Moreover, a deeper understanding of conic sections was useful in practical applications of mathematics, such as studying and constructing dioptras and catoptrics. Therefore, he delved into the construction of tools for accurate conic section design, which he classified as part of his "Mechanics" to distinguish it from his "Mathematics". The first four chapters of his work deal with the conceptual design of curves, while the remaining ten chapters focus on the construction and use of instruments, providing details on construction techniques, materials, and especially the "transfer of motion" from one part of the instrument to another (Dopper, 2014).

The operation of this parabolograph is based on the "equidistant" definition of the parabola, as usually is introduced in upper secondary education today (Figure 2).

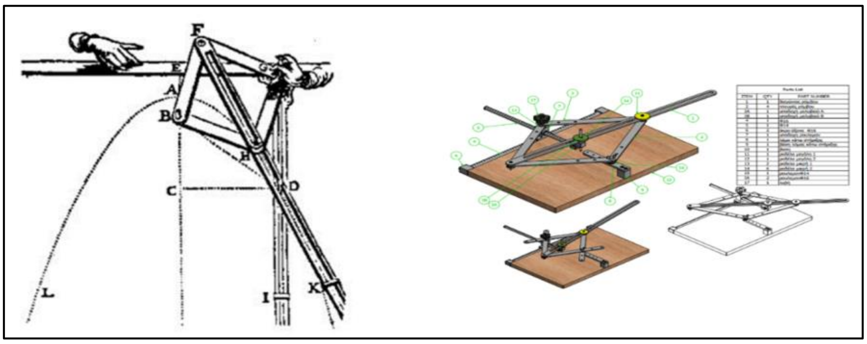

Figure 2: Frans van Schooten's parabolograph

Additionally, the property of the rod FHDK as a tangent line to the traced curve highlights a potential field of slopes for point D (Milici, Plantevin & Salvi, 2022). Triangles BHD and GHD are equal, having HD as a common side, BH and GH as equal sides of the rhombus, and their included angles equal as supplements of the equal angles created by the diagonal and bisector FH of the rhombus. Consequently, the opposite sides DB and DG are equal. Let E be the projection of point B onto the line QR. By placing a Cartesian coordinate system at E of the line QR, with the horizontal axis x'x along the line QR, an attempt will be made to describe the situation algebraically. Applying the Pythagorean Theorem to triangle BCD (Figure 3), we obtain:

$$BD^2 = BC^2 + CD^2 \Rightarrow \cdots \Rightarrow CD^2 = 2DG \cdot BE - BE^2 \Rightarrow |x|^2 = 2(-y) \cdot p - p^2 \Rightarrow$$
$$x^2 = -2py - p^2 \Rightarrow 2py = -x^2 - p^2 \Rightarrow y = \frac{-1}{2p}x^2 - \frac{p}{2}$$

and we now have the algebraic equation of the parabolic curve traced by the machine according to the chosen Cartesian coordinate system. Furthermore, the form of the equation demonstrates its symmetric result with respect to the y'y axis. This "1-1" mechanical input and output, or this "1-1" point-wise geometric transformation of the points on the horizontal line QR into points on the parabola, is captured algebraically as

$$G(x,0) \to D\left(x, \frac{-1}{2p}x^2 - \frac{p}{2}\right)$$

which is a reversible mapping $R^2 \to R^2$ (Pneumatikos, 1974).

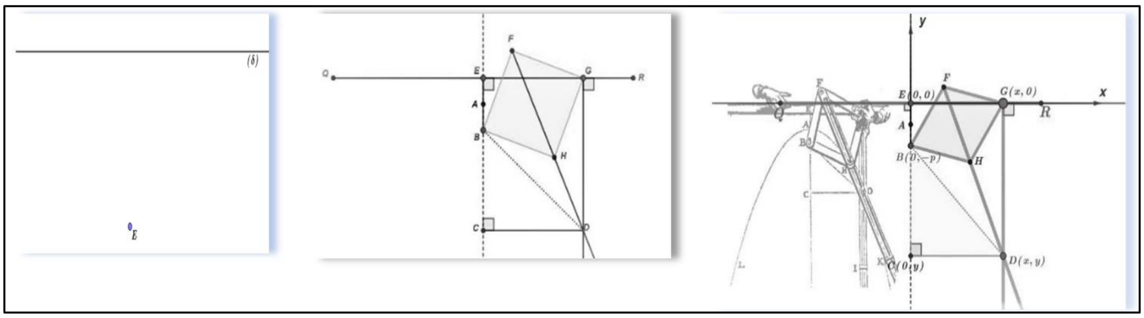

Figure 3: From the "equidistant property" of parabola highlighting the relevant parabolograph

4. Pythagoras mechanism

Christer Bergsten (2015) describes the digital construction of a parabola in a dynamic geometry digital environment, with multiple extensions and applications in the concept and teaching of parabolas. The mechanical construction of this complex mechanism is still in progress and will be presented later, based on the Pythagorean theory of the "parabola" of an equivalent parallelogram (given an angle and a side) with another parallelogram. During its operation, an equivalent square continuously appears within an existing rectangle, while a parabolic arc is simultaneously traced.

The construction of a rectangle with a known side which is equivalent to a square of known sides is carried out according to "Thales" theorem, and according to Propositions I.42 and I.44. But the construction and application of a square with a rectangle of known sides is constructing by Propositions II.14 and VI.8. In Proposition I.43 it is stated that the "complements" of the diagonal of every parallelogram are equivalent (Figure 4). In Proposition I.42, the construction of a parallelogram equivalent to a given triangle is described. Then, in Proposition I.44, the construction of a new equivalent parallelogram is described, of which one side and one angle are known, and it is equivalent to the previous parallelogram. In other words, from a convex planar shape, an equivalent parallelogram is "constructed" (given an angle and a side), using simple means, and it must be "applied" to a specific segment with a specific angle at one of its ends.

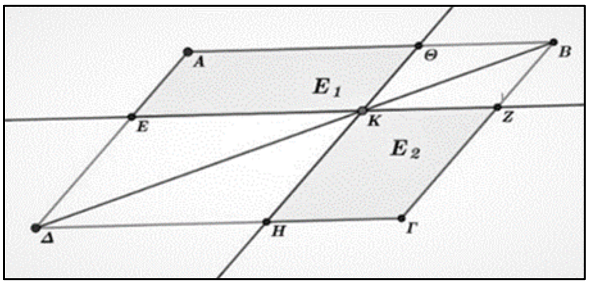

Figure 4: Proposition I.43 of Euclid's Elements and "complements" of α parallelogram

As Proclus states (5th century AD), "Eudemus of Rhodes (4th century BC) attributed to the Pythagoreans the following: When an equivalent parallelogram is applied, then we have a simple "parabola" of an area and not just the construction of a equivalent area. When a parallelogram of smaller area by one square is applied, we have a deficient "parabola" of an area, and when a parallelogram of larger area by one square is applied, we have an excessive "parabola" of an area. The theory of area application constitutes a very important topic in Greek Geometry and serves as a geometric method for solving mixed quadratic equations (Heath, 2001). It is noteworthy that in all these propositions, numerical values are not used, and thus algebraic relationships are derived through Geometry. Proclus states that Eudemus of Rhodes claims that "*the simple application of areas, the application of areas by deficiency, and the application of areas by excess are discoveries of the Pythagoreans*". Subsequent mathematicians adopted these names to use them for the flat curves that resulted from the intersection of cones or cylinders. This mathematical apparatus which could also be considered as an "area measurer" and it will be referred to as the "Pythagoras mechanism".

4.1. The simple Pythagoras mechanism

The fixed and immovable rod AB is articulated vertically with the rods AD and BG (AD = BG) (Figure 5). Points E and H on AB and AΔ respectively are equidistant from A and H due to the placement of a 45° articulation at H, allowing only sliding motion on rod AD (prismatic joint). The rod EZ is articulated at a 90° angle and prismatically attached to AB, allowing it to slide while maintaining its perpendicularity. Rod KZH is vertically articulated at points K, Z, and H, and it is also prismatically attached to the corresponding rods at these points. Additionally, the diagonal rod AK of the rectangular ABHK is articulated both rotationally and prismatically at A and K. At the intersection point Ω of the side of square AEZH and the diagonal AK, a stylus has been articulated rotatably and prismatic. The side AH of square AEZH plays the role of an independent variable, as point A is fixed, and point H (input of the mechanism) slides on rod AΔH. Its movement activates the entire mechanism, simultaneously moving the rods EΩZ, HZK, and AΩK parallel to their initial positions, thus tracing a curve from the stylus to point Ω. In each position of cursor H, the variable (in terms of its dimensions) square AEZH is equivalent to the variable rectangle ABΓΔ (with a fixed side AB and an imaginary side ΓΩΔ) due to diagonal AΩK in rectangle BAHK. In other words, "the square of the variable number (AH) is equal to the product of the variable number (AΔ) and the fixed number (AB)." By suitably introducing a Cartesian coordinate system at point A, with line AΔH as the x'x axis and line AB as the y'y axis, the algebraic equation for the coordinates of point Ω (x, y) is derived as $x^2 = (AB) \cdot y$, representing a parabola in an algebraic framework (register) with "latus rectum" equal to (AB). Moreover, in the special case where (AB) = 1, it serves as a mathematical mechanism for extracting square roots (de-squaring mechanism).

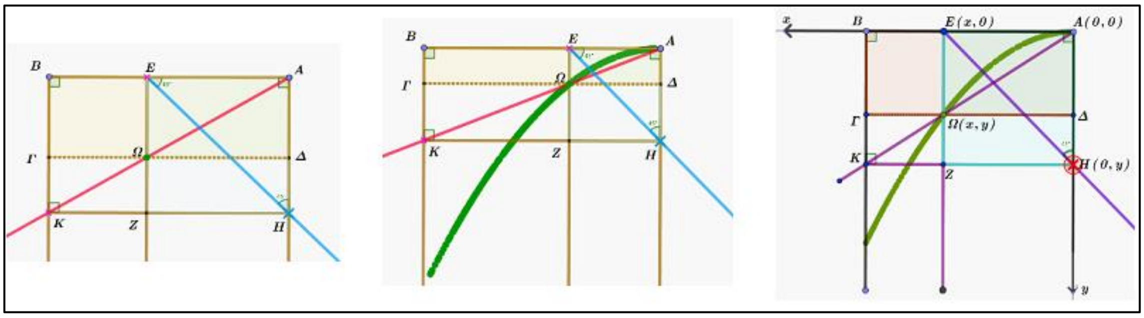

Figure 5: Synthesis & analysis of simple “Pythagoras” parabolograph (https://www.geogebra.org/m/uhdzghyu)

4.2. The complex Pythagoras mechanism

This mechanism also includes the symmetrical part of the previous mechanism concerning the axis of sliding of the cursor H (Figure 6). Specifically, in the extension of

BA by the same length AB', a rod is hinged vertically to B' with the ability to slide on BAB'. At the cursor H is hinged a rod HE' perpendicular to HE with a prismatic joint on H and on this rod as well as a prismatic joint in E΄ of BB΄. This results in the isosceles right triangle HAE', which is equal to triangle HAE for any position of points H and H'. The extension of KH is articulated perpendicularly and prismatically at K' of B'Γ', and the perpendicular of BB' at E' is articulated perpendicularly (prismatically on B'B'). It is then articulated perpendicularly and prismatically at Z' of KK'. Finally, at the fixed-point A, the diagonal of the rectangle AHK'B' is articulated rotationally which is articulated rotationally and prismatically at point K'. This diagonal is also articulated prismatically at point *Ω'*on the side E'Z' of the square AHZ'E', where a stylus is also placed.

By moving the cursor H, the parabolic arc ZAZ' is traced. For any position of point H, the angle AHE = 45° due to the joint, and quadrilateral AHEZ is a square. Rods HE and H'E' are always perpendicular to each other due to the joints, and quadrilateral AE′Z′H is also a square, equal to AEZH and symmetric with respect to rod AH. The axial symmetry of the mechanism with respect to rod AH is evident, as well as the symmetry of the curves traced by the two stylus Ω and Ω΄.

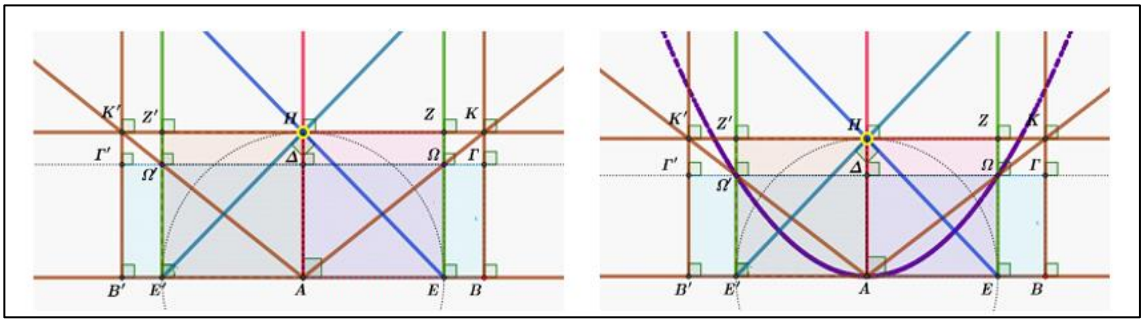

Figure 6: Synthesis of complex "Pythagoras" parabolograph
(https://www.geogebra.org/m/vk4hawxy)

Furthermore, by continuously reducing the distance AH, it becomes apparent through the apparatus that at point A, the curve (although it may be difficult to meet the two points constructively) is "smooth" due to the "application" of the two squares at A, having collinear sides AE and AE', and, therefore, right angles at A, making A the vertex of the parabola and axis of symmetry the line AH.

5. Mechanical confirmation of the parabola definitions equivalence

The following is the process of proving the equivalence of the definitions on which the operation of these parabolographs is based in a mechanical environment. Specifically, assuming one of the three mechanisms given, there is an attempt, in a dynamic geometry environment, to articulate rods in a way that does not hinder the operation of the first mechanism and at the same time, the second mechanism also emerges, which works dependently and tracing exactly the same curve and without being able to use its "input", while its "output" is identified with the "output" of the first mechanism.

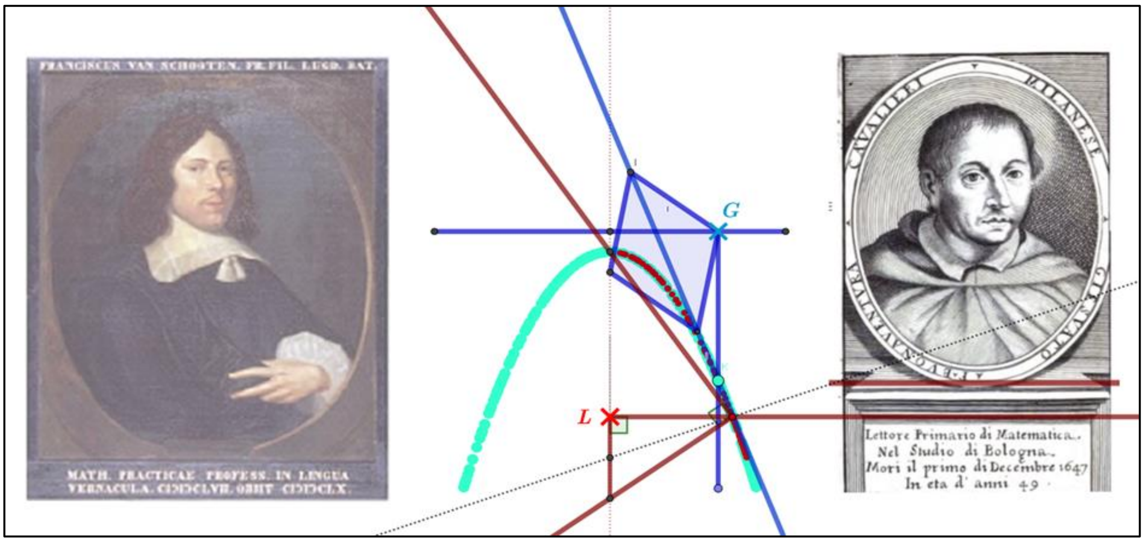

Figure 7: van Schooten and Cavalieri meeting (https://www.geogebra.org/m/nnjcunyv)

5.1. From "van Schooten" to "Cavalieri" mechanism

Starting from the van Schooten's parabolograph due Pythagorean theorem, as well as from the definition of "equal distances" by Pappus (Heath, 2001), which is evidently included in its structure (Figure 8), it is derived:

$$(\mathrm{FV})^2 + (\mathrm{VD})^2 = (\mathrm{DF})^2 = (\mathrm{DG})^2 \Leftrightarrow [(\mathrm{AV}) - (\mathrm{AF})]^2 + (\mathrm{VD})^2 = (\mathrm{EV})^2 = [(\mathrm{AV}) + (\mathrm{AE})]^2 \Leftrightarrow (\mathrm{AV})^2 - 2(\mathrm{AV})(\mathrm{AF}) + (\mathrm{AF})^2 + (\mathrm{VD})^2 = (\mathrm{AV})^2 + 2(\mathrm{AV})(\mathrm{AF}) + (\mathrm{AF})^2 \Leftrightarrow (\mathrm{VD})^2 = 4(\mathrm{AV})(\mathrm{AF}) = (\mathrm{AV}) \cdot 4(\mathrm{AF}) = (\mathrm{AV}) \cdot 2(\mathrm{EF}) = (\mathrm{AV}) \cdot 2(\mathrm{VU}) \Leftrightarrow (\mathrm{VD})^2 = (\mathrm{AV}) \cdot 2(\mathrm{VU}) \Leftrightarrow \frac{(\mathrm{VD})^2}{(\mathrm{AV})} = 2(\mathrm{VU})$$

and that is the equation of parabola according to Apollonius (Stamatis, 1975). A characteristic property of the tangent line at each point D of the parabola is to form a "subnormal" VU that is equal to its "semi-latus rectum". Therefore, the "normal" DU defines the "subnormal" VU, which is equal to the " semi latus rectum " p, that is, the distance (FE). Thus: $(\mathrm{VD})^2 = (\mathrm{AV}) \cdot 2\mathrm{p} = (\mathrm{AV}) \cdot$ "latus rectum" $= (\mathrm{TV}) \cdot$ "semi-latus rectum" or equivalently, the square of side equal to the "ordinate" (according to Apollonius) VD, is equivalent to the rectangle of sides equal with "semi-latus rectum" EF=VU and the "sub-subtangent" TV=2AV. The constant squaring is evident from the van Schooten parabolograph, of the rectangle with these specific sides for any position of the cursor G.

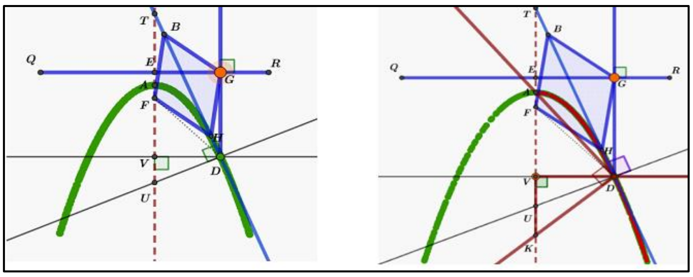

Figure 8: From "van Schooten" to "Cavalieri" parabolograph (https://www.geogebra.org/m/vuabfkjr)

Mechanically, by assembling on the van Schooten mechanism at V, a prismatically articulated (on the rod EF) rod VK with length of a "latus rectum," where at K and A, two rods are articulated rotatably and prismatically, which then articulate rotationally at D of the van Schooten parabolograph. The van Schooten parabolograph (through its structure) consistently produces the relation $(\mathrm{VD})^2 = (\mathrm{AV}) \cdot 2(\mathrm{VU})$ or equivalently, in any position of the cursor G, $(\mathrm{VD})^2 = (\mathrm{AV}) \cdot (\mathrm{VK})$. The linking for the Cavalieri parabolograph, which is emerged from the van Schooten one, does not require a priori orthogonally articulating the rods DA and DK at D because orthogonality arises through the relation $(\mathrm{VD})^2 = (\mathrm{AV}) \cdot (\mathrm{VK})$, which defines a right angle at D according to the first lemma of Pappus regarding the fifth Book of Apollonius (Stamatis, E. 1976, p. 157). Also, in the mechanism of van Schooten, the diagonal rod DT of the rhombus has the direction of the tangent line at D of the drawn parabola, and the characteristic property of the "subcanonical" VU is that it is equal to a "semi-latus rectum" for every position of D. Thus, the way to attach the tangent line to every point D emerges for the Cavalieri mechanism as well. It suffices to articulate, in a rotating and sliding manner, a rod connecting point D with the midpoint U of the (fixed length equal to the "latus rectum") rod VK, and then to articulate at D a perpendicular rod with DU which implements the tangent to the parabola at D.

Finally, from the definition of the parabola through equal distances from a point and a line, the definition of Apollonius' parabola arises through the intersection of a cone by a plane parallel to its generator.

5.2. From "Cavalieri" to "van Schooten" mechanism

Starting from the Cavalieri parabolograph, the corresponding focus F and the directix (δ) are placed on the plane, knowing that the "latus rectum" of the parabola is equal to the length of segment VK (Figure 9). A parabola is constructed because for every position of point D, it results from the permanently right triangle ADK and according to Proposition VI.8 (Exarhakos, 2001).

$$(\mathrm{VD})^2 = (\mathrm{AV}) \cdot (\mathrm{VK}) \Rightarrow (\mathrm{VD})^2 = 2(\mathrm{AV})(\mathrm{VU}) = 2(\mathrm{AV}) \cdot (\text{semi-latus rectum}),$$

where U is the midpoint of segment VK and trace of the "normal" DU.

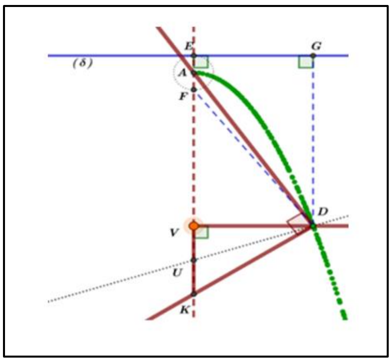

Figure 9: Start of construction of "van Schooten" through "Cavalieri" parabolograph

However, this relationship constitutes the equivalent definition of the parabola according to Apollonius (Stamatis, 1975). In other words, the square of (DV) is equal to the product of (AV) multiplied by the "latus rectum". Similarly, the square of a side equal to VD is equivalent to the rectangle with sides "semi- latus rectum" EF=VU and AV. The continuous squaring of the rectangle with these specific sides is evident for any position of the cursor V. From the equation:

$$(\mathrm{VD})^2 = 2(\mathrm{AV}) \cdot 2(\mathrm{AF}) = 4(\mathrm{AV})(\mathrm{AF})$$

by adding a suitable quantity to both members, it follows that:

$$(\mathrm{AV})^2 + (\mathrm{AF})^2 - 2(\mathrm{AV})(\mathrm{AF}) + (\mathrm{VD})^2 = (\mathrm{AV})^2 + (\mathrm{AF})^2 - 2(\mathrm{AV})(\mathrm{AF}) + 4(\mathrm{AV})(\mathrm{AF}) \Leftrightarrow$$
$$[(\mathrm{AV}) - (\mathrm{AF})]^2 + (\mathrm{VD})^2 = (\mathrm{AV})^2 + (\mathrm{AF})^2 + 2(\mathrm{AV})(\mathrm{AF}) = [(\mathrm{AV}) + (\mathrm{AF})]^2 =$$
$$[(\mathrm{AV}) + (\mathrm{AE})]^2 = (\mathrm{VE})^2 \Leftrightarrow (\mathrm{FV})^2 + (\mathrm{VD})^2 = (\mathrm{VE})^2 = (\mathrm{d}\,(\mathrm{V}, (\delta))^2$$

From the triangle FVD, it follows that $(\mathrm{FV})^2 + (\mathrm{VD})^2 = (\mathrm{FD})^2$, and from the last two equations,

$$(\mathrm{FD})^2 = (\mathrm{VE})^2 \Rightarrow (\mathrm{FD}) = (\mathrm{VE}) = (\mathrm{DG}) \Rightarrow \mathrm{d}(\mathrm{D}, \mathrm{F}) = \mathrm{d}(\mathrm{D}, (\delta))$$

meaning that any point D on the parabola traced by Cavalieri parabolograph, in addition to the fundamental property of the mechanism emerging Apollonius' definition, has the property of equal distances from the point F and the line (δ) that constitutes the definition of Pappus. Furthermore, from the characteristic property of the points on the perpendicular bisector of segment FG, to which D belongs, van Schooten mechanism gradually emerges. Point G is shifted onto line (δ), and point D belongs to the perpendicular bisector of the corresponding segment FG (due to the definition of equidistance) as well as the perpendicular line to line (δ) at point G.

The property of the perpendicular bisector is possessed by the diagonals of any rhombus, regardless of whether its angles change. For any position of point D in the Cavalieri mechanism, there exists an "invisible" rhombus that establishes the property of D belonging to the perpendicular bisector DT of FG (Figure 10). However, since the sides of the "invisible" rhombus FDGT change at each position of D, a real rhombus FHGB with FG as its imaginary diagonal is introduced into the Cavalieri mechanism. This rhombus has the other diagonal BH as the perpendicular bisector of FG and is collinear with the diagonal DT of the invisible rhombus FDGT. By the equality of triangles FGD and FTG, it follows that the fourth vertex of the "invisible" rhombus is T, and the diagonal DHBT is common to both rhombuses.

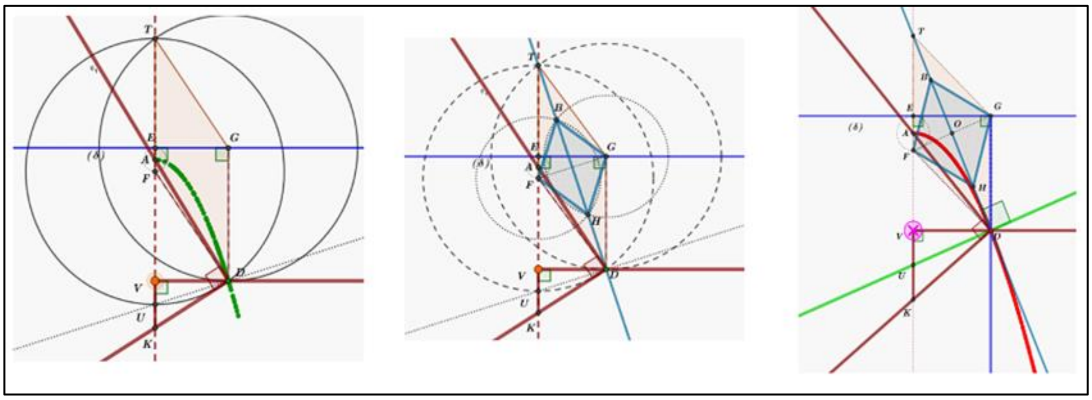

Figure 10: "van Schooten" through "Cavalieri" parabolograph,
until its complete emergence (https://www.geogebra.org/m/dnskbeup)

According to the definition of the "normal" by Leibnitz (Dennis & Confrey, 1995), the tangent line at D of the parabola traced by the Cavalieri mechanism is perpendicular to the normal DU at D of the parabola, where U is the midpoint of segment VK. The tangent of the traced parabola also represents the diagonal line DT of both rhombi, and it has already been proven (Schooten, 1646) that this diagonal is the tangent at any point of the parabola traced by the "emerging" mechanism of Frans van Schooten.

Mechanically, to emerge the "van Schooten" mechanism from the "Cavalieri" mechanism and trace the exact same parabolic curve, the following steps are followed: (a) a rod is articulated at the midpoint U of the fixed-length rod VK and at point D, both prismatically and rotationally, (b) a sliding rod is placed on the directrix, (c) a permanently perpendicular rod DT is articulated rotationally and prismatically at D with respect to UD, (d) a rhombus is articulated rotationally at Focus F and prismatically and rotationally at G on the rod EG, (e) the rod DT is articulated prismatically with the vertices B and H of the rhombus, (f) a rod is articulated prismatically at points D and G, which is essentially perpendicular to the rod that represents the directrix. Therefore, from the definition of the of Apollonius, the iso-distant definition of the parabola is derived. Moreover, through this process, the Cavalieri parabolograph highlights the tangent of the parabolic arc traced at any point D.

5.3. From "Pythagoras" to "Cavalieri" mechanism

The complex Pythagoras mechanism can *square* two equal rectangular parallelograms with equal, constant, and co-linear sides AB and AB'. For successive values of the other sides BΓ and B'Γ', the diagonals AΓ and AΓ' intersect the sides of the equivalent squares AEΘΔ and AE'Θ'Δ, respectively, at points Ω and Ω', which, in addition to being symmetric with respect to the line AΔ, are also points of a parabola with "latus rectum" equal with (AB) (Figure 11). Due to the bisecting of ΩΩ' and its parallelism with BB', it follows, according to Proposition 5 of Book II of Apollonius (Stamatis, 1976), that the top diameter and vertex of the parabolic arc are AΔ and A respectively. Since the traced parabolic arc is symmetric with respect to the diameter AΔ, it follows that AΔ is the axis of the parabola, and point A, in addition to being the vertex of the arc, is also the vertex of the corresponding parabola.

Articulated a rod on points Ω and Ω' and articulating it, the point H is defined on the axis of the parabola. By prismatically placing a rod HK on the axis of parabola with one end at H, with a length equal to the fixed side AB of the rectangle, and then articulating two rotating rods at points-stylus Ω and Ω' and rotating-prismatically with the other end K of HK, the double Cavalieri parabolograph is emerged. Indeed, the variable common side AH of the rectangles ABZH and AB'Z'H is collinear with the segment HK, for which (HK) = (AB) = "latus rectum" and the segments ΩH and Ω'H have a length equal to the side of the equal squares AEΘΔ and AE'Θ'Δ. Through the property of the diagonal of a parallelogram to define equal triangles (Proposition I.34), it is proven that the rectangle is equal to the rectangle, i.e., $(\mathrm{AE})^2 = (\mathrm{AB}) \cdot (\mathrm{BZ}) \Leftrightarrow (\mathrm{H}\Omega)^2 = (\mathrm{HK}) \cdot (\mathrm{HA})$. According to the 1st lemma of Pappus, related to the 5th Book of Apollonius (Stamatis,1976), it is demonstrated that the triangles AΩK and AΩ'K are right triangles at Ω and Ω', respectively. Therefore, the perpendicularity of the rods of the double Cavalieri parabolograph at points Ω and Ω' is imposed by the Pythagoras parabolograph, and there is no need for right-angle joints at these points.

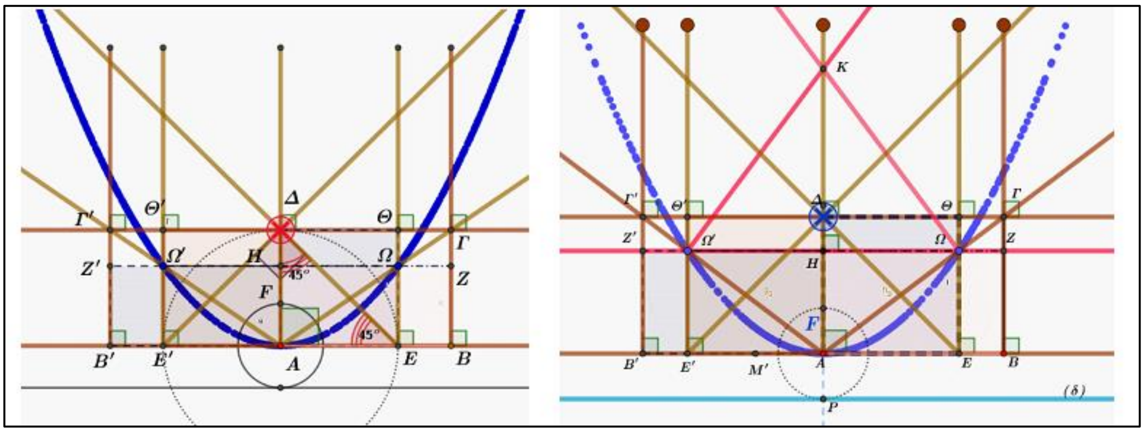

Figure 11: Cavalieri through Pythagoras parabolograph
(https://www.geogebra.org/m/qhwpxyvp)

It has already been proven that for an arbitrary point on the parabola, the segment connecting the Focus and the projection of that point onto the directrix is bisected perpendicularly by the tangent at that point, or conversely, the perpendicular bisector to this segment tangentially touches the respective point on the parabola (Ntontos, 2019). By constructing the Focus F and the directrix (δ), and based on this property, the equality of triangles FAM and MEΣ is proven, resulting in the point M, where the tangent at Ω intersects the tangent at vertex A of the parabola, always being the midpoint of the segment AE, where E is the projection of point Ω onto the tangent BB'. Therefore, the midpoints M and M' of segments AE and AE' are sufficient to construct tangents at points Ω and Ω' of the parabola (Figure 12). For the tangent line to be articulated at (moving) point Ω of the "Pythagoras" mechanism's parabola, Ω must be connected to the midpoint of AE, as the positions of Ω and E are continuously changing. It is sufficient to extend the rod ΘE of the square AEΘΔ and articulate it prismatically at point Σ of the fixed rod (δ). Then, using the "imaginary" diagonal FΣ, the "van Schooten rhombus" is articulated, where the diagonal rod ΩT will be the tangent line at point Ω of the parabolic curve. Through this addition, the tangent line at every point Ω and Ω′ of Pythagoras parabolograph can also be articulated. It has been proven for every parabola that it constitutes the "iso-subnormal" curve, and the right-angled triangle NΩH (of the normal NΩ and subnormal NH) are always equal with the right-angled triangle FPΣ (Figure 11). This permanent equality of triangles makes N the midpoint of rod HK, which has a length equal to the latus rectum AB of the traced parabola. Therefore, the medians ΩN and Ω'N of triangles ΩHK and Ω'HK are the "normal" lines at points Ω and Ω′ of the parabola, and therefore are perpendicular to the corresponding tangent lines. By already placing the tangent rods at points Ω and Ω′ and with the mediation of the van Schooten rhombus, the medians ΩN and Ω'N are always perpendicular to the already tangent lines at Ω and Ω′ without the appropriate articulation of a "right angle".

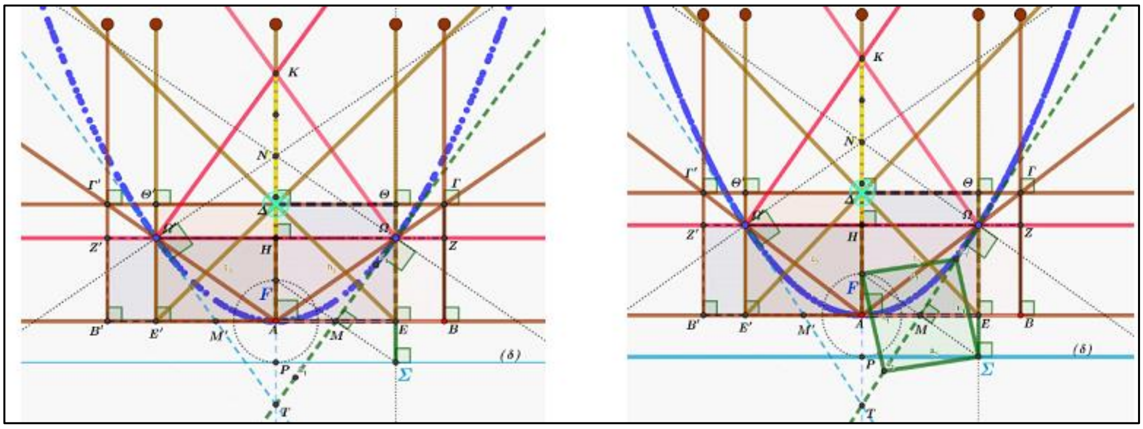

Figure 12: Appearance of the tangent to the traced parabola
(https://www.geogebra.org/m/qhwpxyvp)

Ultimately, starting from the complex Pythagoras parabolograph of and articulating rods properly, and with the mediation of the "van Schooten parabolograph rhombus", the Cavalieri parabolograph emerges mechanically along with its tangent line. From definition of parabola as an equivalent square transformed into a rectangular parallelogram with one fixed side, the definition of Apollonius parabola is derived.

5.4. From "Cavalieri" to "Pythagoras" mechanism

In double Cavalieri parabolograph, rod BB' with a length twice of the "latus rectum" HK is placed perpendicular to the axis AK, with the vertex A of the mechanism as its midpoint (Figure 13). Then, perpendicular rods with BB′ are articulated at points B and B', intersecting rod ΩΩ' of the mechanism at points Z and Z', where they are articulated prismatically. Furthermore, the perpendicular rods BZ and B'Z' are articulated prismatically and rotationally at points Γ and Γ' with the rods AΩ and AΩ' of the Cavalieri parabolograph. Due the equality of triangles ABΓ and AB'Γ' and through the inverse theorem of Thales, the rod ΓΓ' articulated at points Γ and Γ' is parallel to rods ZZ' and BB' (Figure 13). The segments ΑΓ and ΑΓ' are diagonals of the parallelograms ΑΒΓΔ and ΑΒ'Γ'Δ and divide them into equal triangles. Joining rods at points Ω and Ω' parallel to the sliding axis of the Cavalieri parabolograph, rectangles ΑΕΩΗ and ΑΕ'Ω'Η are also divided into equal triangles, just like rectangles ΖΩΘΓ and Ζ'Ω'Θ'Γ. Therefore, rectangles ΒΕΖΩ and Β'Ε'Ω'Ζ' are equivalent to rectangles ΩΘΔΗ and Ω'Θ'ΔΗ, respectively. It is evident that (ΑΒΖΗ) = (ΑΕΘΔ) = (ΑΒ'Ζ'Η) = (ΑΕ'Θ'Δ), meaning (ΑΒ)(ΒΖ) = (ΑΕ)(ΕΘ) = (ΑΒ')(Β'Ζ') = (ΑΕ')(Ε'Θ') (∗). In the double Cavalieri parabolograph, according to Proposition VI.8, it follows that $(\mathrm{H\Omega})^2 = (\mathrm{H\Omega'})^2 = (\mathrm{AH})(\mathrm{HK}) \Rightarrow (\mathrm{AE})^2 = (\mathrm{AE'})^2 = (\mathrm{BZ})(\mathrm{AB}) = (\mathrm{B'Z'})(\mathrm{AB'})$ (∗∗). From the relationships (∗) and (∗∗) it follows that $(\mathrm{AE})(\mathrm{E\Theta}) = (\mathrm{AE})^2$and $(\mathrm{AE'})(\mathrm{E'\Theta'}) = (\mathrm{AE'})^2$. Consequently, (ΑΕ) = (ΕΘ) and (ΑΕ') = (Ε'Θ'), which means that the rectangles ΑΕΘΔ and ΑΕ'Θ'Δ are squares, and due to (∗) the rectangles ΑΒΖΗ and ΑΒ'Ζ'Η are "squared" as well. Therefore, the Pythagoras parabolograph, which "applies" an equivalent square to a rectangle of fixed side, and simultaneously the points Ω and Ω' trace the same parabola already drawn by the original double Cavalieri parabolograph. By moving the cursor Η of Cavalieri parabolograph, the parabola is traced by the stylus at Ω and Ω', and the cursor Δ of the mechanism follows, just Pythagoras mechanism.

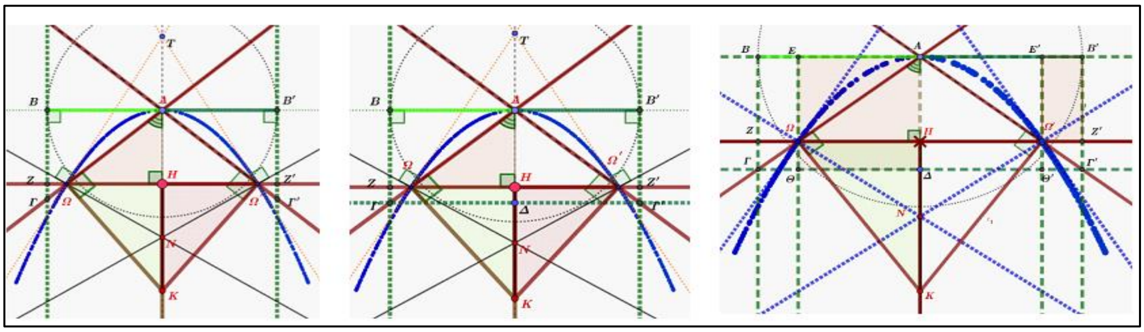

Figure 13: "Pythagoras" from "Cavalieri" parabolograph
(https://www.geogebra.org/m/pxdfgkvr)

Regarding the tangent line at points Ω and Ω' of the parabolic curve, which is simultaneously traced by both mechanisms, the "normal" lines ΝΩ and ΝΩ′ (medians of triangles ΩΗΚ and Ω'ΗΚ) in Cavalieri parabolograph define the tangent straight lines at points Ω and Ω' of the parabola, which are traced by both mechanisms. These tangent lines must pass through the midpoints of the variable segments ΑΕ and ΑΕ' without the mediation for van Schooten's rhombus (Ntontos, 2019), thanks to the right-angled joints of the tangent lines at points Ω and Ω' of the parabolic curve.

Ultimately, starting from the double Cavalieri parabolograph and suitably connecting certain rods, the mechanical construction of the complex Pythagoras parabolograph emerges along with its tangent at the point-stylus. From the definition of "Apollonius parabola", the Pythagorean definition of parabola is deduced.

5.5. From "Pythagoras" to "van Schooten" mechanism

In the Pythagoras mechanism, the rods connecting points Ω and Ω' with the midpoints M and M' of the segments AE and AE', respectively, constitute the corresponding tangent lines. However, to attach them to the mechanism, the focus F and the directrix (δ) must be positioned, knowing already that the "latus rectum" of parabola has the same length as the fixed rod AB of the mechanism. Next, the rod ΩE must be extended by a distance equal to (AF) and hinged prismatically at point Σ of the directrix (δ) (Figure 14). Additionally, the perpendicular bisector of the imaginary segment FΣ must be positioned for any position of the moving points Ω and Σ. This can be achieved by connecting a rhombus at points F and Σ, and then attaching a straight rod to the other two vertices.

The *flexibility* of the rhombus ensures that it remains unchanged in terms of the properties of its diagonals, as the cursor Δ of the Pythagoras mechanism moves and point Σ moves along the directrix (δ). The vertex F rotates on itself, the diagonal ΩT is perpendicular bisector to the *imaginary* diagonal FΣ, and the equality of triangles FAM and ΣEM permanently makes the variable segments AM and ME equal. At last, the construction of van Schooten parabolograph was revealed, which is based on the definition of equal distances by Pappus. To trace the left branch of the parabola and by van Schooten mechanism, one only needs to follow the same process.

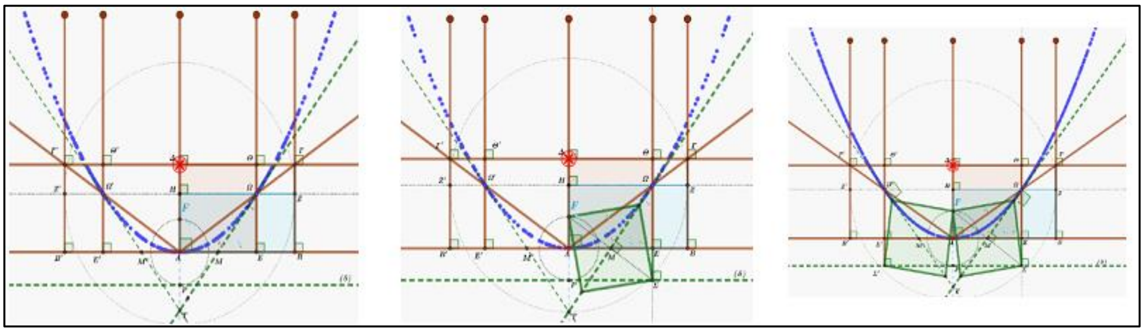

Figure 14: "van Schooten" from "Pythagoras" parabolograph
(https://www.geogebra.org/m/s4fjekn8)

Finally, starting from the complex Pythagoras parabolograph and suitably articulating some rods, we obtain the Frans van Schooten parabolograph, which follows the movement of the first one as the cursor Δ of Pythagoras moves. By following the movement of its cursors Σ and Σ', it produces the same output-stylus, thus tracing the same parabolic curve. In other words, from the Pythagorean definition of the parabola-application of an equivalent square in a rectangular parallelogram with one side fixed, the equidistant definition of Pappus arises.

5.6. From "van Schooten" to "Pythagoras" mechanism

In the midpoint A of distance FE from Focus F to the directrix (δ) of the parabolograph van Schooten, a straight rod parallel to the directrix is placed. Then, with center A and a radius twice the distance FE = "semi-latus rectum" points B and B' are selected on the rod (Figure 15). Perpendicular rods with BB′ are placed at points B and B', and an oblique rod hinged rotationally at fixed point A and rotationally prismatically at stylus D of diagonal DT, which hinges rotationally-prismatically at point Γ on the perpendicular BB' at B. A rod perpendicular to the axis of the parabola is hinged at point Γ, which is hinged prismatically at point Γ' with the perpendicular rod at B' of BB'. Through appropriate joints, the segment added to the van Schooten mechanism follow its motion. According to Proposition I.43 of Euclid's Elements, for any position of cursor G, the rectangle ABZK is equivalent to the rectangle ΑΛΘΔ. It has been shown that in the van Schooten parabolograph, for any position of point D, the metric relationship holds: $(\mathrm{DK})^2 = 2(\mathrm{FE}) \cdot (\mathrm{AK}) = 2(\mathrm{FE}) \cdot (\mathrm{D\Lambda})$. Therefore, $(\mathrm{DK})^2 = \text{latus rectum} \cdot (\mathrm{BZ}) = (\mathrm{AB}) \cdot (\mathrm{BZ})$ which implies that the rectangle ABZK is equivalent and with a square side of DK = ΘΔ = ΑΛ. Consequently, the rectangle ΑΛΘΔ must be α square equivalent to rectangle ABZK and "applied" on side AB and angle A for any position of cursor G. In other words, stylus D traces a parabolic arc while simultaneously revealing the side of a square equivalent to a rectangle with sides equal to the "latus rectum" and the distance of D from the parabola's tangent at its vertex.

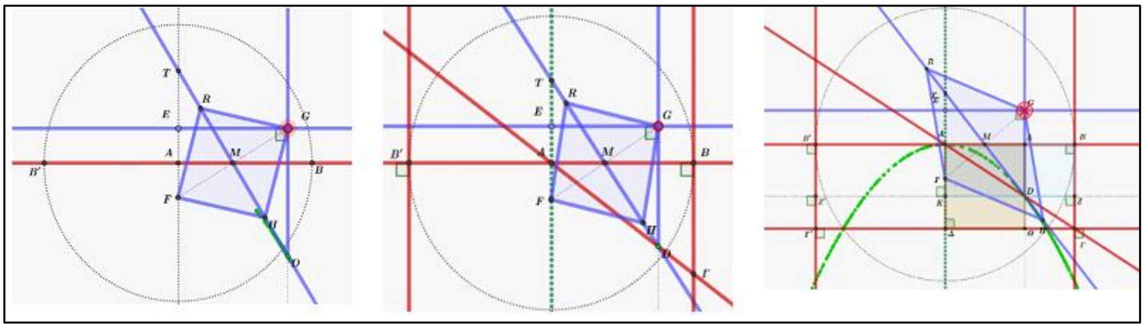

Figure 15: "Pythagoras" from "van Schooten" parabolograph
(https://www.geogebra.org/m/cuqs8vks)

Finally, starting from the van Schooten parabolograph and properly articulating some rods, the Pythagoras parabolograph emerges, which follows the motion of the first one as the cursor G of van Schooten moves, the cursor D of Pythagoras follows the same output-stylus and traces the same parabolic curve. In other words, from the *equidistant* definition of Pappus, the Pythagorean definition of the parabola-application of an equivalent square on a parallelogram with one of its sides fixed is derived.

6. Conclusion

In this article, three parabolographs were initially presented, each constructed based on a different definition of the parabola. Then, by composing each mechanism on another without hindering with the operation of the first, their mechanical equivalence is confirmed as well as the equivalence of the corresponding definitions of the parabola.

According to the theory of semiotic mediation, the exploration of a machine should start from a physical manipulation and then progress to a conceptual and mathematical understanding of the artifact. It is then proposed in this paper that with a basic knowledge of engineering design and the use of dynamic geometry software, the reverse process of designing and synthesizing each parabolograph on top of another one can be implemented so that the operation of the first one is not hindered by the second one following it and they have the same result in their common *output*.

This pedagogical proposition is daring and anticipates the introduction of machines in Mathematics education through a S.T.E.M. approach. It can be adapted for different purposes at various levels: for high school students, Mathematics or Mechanical Engineering undergraduates, as well as for professional mathematicians.

Although there is much work to be done in this direction, this manuscript could potentially serve as a step for the use of mathematical machines in interdisciplinary Mathematics education.

Conflict of Interest Statement
The authors declare no conflicts of interest.

About the Authors

The author is a Mathematician at the 9th Lyceum of Patras and a PhD candidate at the University of Patras with the supervising Professor of Mathematics Ms. Koleza Eugenia. He has published articles with collaborators at conferences of the Hellenic Mathematical Society on experimental Mathematics and STEM education. The subject of research is analysis, simulation, construction of mathematical machines and introduction to secondary education.

References

Bennett, J. (2011). Early modern mathematical instruments. *Isis: an International Review Devoted to the History of Science and Its Cultural Influences*, *102*(4), 697–705. Retrieved from: https://doi.org/10.1086/663607

Bergsten, C. (2019). Beyond the Representation Given – The Parabola and Historical Metamorphoses of Meanings. *In PME, International Group for the Psychology of Mathematics Education,* (pp. 1-264). Bergen: Bergen University College. Retrieved from: https://www.researchgate.net/publication/282879109_Beyond_the_representation_given_The_parabola_and_historical_metamorphoses_of_meanings

Cajori, F. (1893). *A History of Mathematics*. The Mac Millan Company. Retrieved from: http://www.public-library.uk/dailyebook/A%20history%20of%20mathematics%20(1894).pdf

Cavalieri, B. (1632). *Lo specchio ustorio, overo, Trattato delle settioni coniche, et alcuni loro mirabili effetti intorno al lume, caldo, freddo, suono, e moto ancora.* Bologna. Retrieved from https://play.google.com/books/reader?id=ZX64AAAAIAAJ&pg=GBS.PA186&hl=el

Cavalieri, B. (1635). *Geometria indivisibilibus continuorum nova quadam ratione promota.* Bologna. (1st ed.). Retrieved from: https://ia600702.us.archive.org/32/items/ita-bnc-mag-00001345-001/ita-bnc-mag-00001345-001.pdf

Chang, W. T., & Yang, D. Y. (2020). A note on equivalent linkages of direct-contact mechanisms. *Robotics*, *9*(2). Retrieved from: https://doi.org/10.3390/ROBOTICS9020038

Dennis, D., & Confrey, J. (1995). Functions of a curve: Leibniz's original notion of functions and its meaning for the parabola. *The College Mathematics Journal, 26(2),* 124-131.

Dopper, J. G. (2014). A life of learning in Leiden. *The mathematician Frans van Schooten (1615-1660)* (Doctoral dissertation). University Utrecht, Netherlands. Available from Utrecht University Repository. Retrieved from: https://dspace.library.uu.nl/handle/1874/288935

Exarhakos, G. (Ed.). (2001). *Euclid "Elements"* (1st ed., Vol. 1). Athens: Center for Science Research and Education. Retrieved from: https://commonmaths.weebly.com/uploads/8/4/0/9/8409495/tomos-1.pdf

Heath T. L. (2001). *History of Greek Mathematics, From Aristarchus to Diophantus* (cf. Aggelis, A., Vlamou, E., Grammenos, T., Spanou A.). Athens: Κ.Ε.ΕΠ.ΕΚ. (1921 is the year of original publication)

Malet, A. (1996). From Indivisibles to Infinitesimals: Studies on Seventeenth Century Mathematizations of Infinitely Small Quantities. *University of Barcelona, 89(1),* 131–132

Milici, P., Plantevin, F., & Salvi, M. (2022). A 3D-printable machine for conics and oblique trajectories. *International Journal of Mathematical Education in Science and Technology*, *53*(9), 2549–2565. Retrieved from: https://doi.org/10.1080/0020739X.2021.1941366

Ntontos, G. (2019). *Machines for generating parabolic arcs and their application in teaching practice* (Master's Thesis). University of Patras, Patras. Retrieved from: https://nemertes.library.upatras.gr/items/4ca4e2e9-7cc6-45b1-99a2-297229060fea

Pneumatikos, N. (1974). *Geometry Supplement.* (1st). Athens. Retrieved from: https://drive.google.com/file/d/0B9uh0VymSVrpd1A1YnM2ZVN5Mlk/view?resourcekey=0-wZEjD1Hhl_q8V0a8DZaVVQ

Schooten, F. V. (1646). *De organica conicarum sectionum in plano descriptione, tractatus: Geometris, opticis; præsertim veró gnominicis & mechanicis utilis*. Leyden. Retrieved from https://ia904703.us.archive.org/22/items/ita-bnc-mag-00001383-001/ita-bnc-mag-00001383-001.pdf

Stamatis, E. (1975). *Apollonius Conics* (Vol. 1). Athens: Technical Council of Greece. Retrieved from: https://drive.google.com/file/d/1UIC2hzYAFDGLbsGtX6hBTCJNTMyC_AoG/view?pli=1

Stamatis, E. (1976). *Apollonius Conics* (Vol. 4). Athens: Technical Council of Greece. Retrieved from: https://drive.google.com/file/d/1b2FeEvalmJKPtX0zU2RPJCdss3VNbX3_/view

Stamatis, E. (1976). *Apollonius Conics* (Vol. 2). Athens: Technical Council of Greece. Retrieved from: https://drive.google.com/file/d/1h19X4qpUUW0E0-UJWQO4bciBNdYIGbae/view