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PYTHAGOREAN PARABOLOGRAPH FOR REGIONS' QUADRATURE

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Abstract:
In this paper an attempt is made to design a mathematical machine based on the Pythagorean theory of fitting an equivalent parallelogram (given angle and side) with another parallelogram. An equivalent square or rhombus is constructed given a rectangle or a parallelogram respectively. It is then proved that this mechanism can simultaneously draw a parabolic arc. The parameters of this mathematical machine are checked. In each case, an attempt is made to find the focus and the directrix of the corresponding parabola. The important help of the GeoGebra software for the initial mechanical design of the machine is highlighted, but also the important differences of this simulation from the physical construction.

Keywords: parabola, parabolograph, quadrature of regions, artefacts, mathematical machine

1. Introduction

The construction of a rectangle with known one side, equivalent to a given square of known side, is carried out with a ruler and compass and in accordance with Thales' theorem, and in accordance with Propositions I.42 and I.44 of the Euclid's "Elements". However, the construction (and the fitting) of an equivalent square with a rectangle of known sides is carried out according to Proposition II.14 or else according to Proposition VI.8.

In Proposition I.43 of Euclid's "Elements", it is stated that the "complements" of the diagonal of any parallelogram are equivalent (Exarhakos & Dziachristos, 2001, pg.

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194). Specifically, from a random point K on the diagonal of a rectangle $AB\Gamma\Delta$, lines parallel to its sides have been drawn (figure 1). The parallelograms $A\Theta KE$ and $KZ\Gamma H$ are the "complements" (or "supplements") of the diagonal ΔB and are equivalent to each other for any position of K.

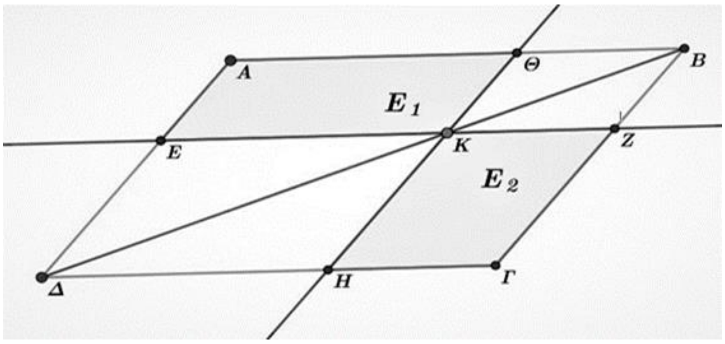

Figure 1: Proposition I.43 of Euclid's "Elements" and "*complements*" of α parallelogram

There are some positions of K such that one of the two complements is a rhombus. Proposition I.42 of Euclid's "Elements" describes the construction of a parallelogram equivalent to a given triangle (Exarhakos & Dziachristos, 2001, p. 195). Then in Proposition I.44 the construction of a new equivalent parallelogram whose one side and one angle are known and is equivalent to the previous parallelogram and to the given triangle is described. That is, starting with a planar convex shape, an equivalent parallelogram (given its angle and side) is "constructed" by simple construction techniques. The new equivalent parallelogram must be fitted to a given segment with a given angle at one end of this segment.

The theory of fitting regions is a very important topic for Greek Geometry and is a geometric method for solving mixed quadratic equations (Heath, 2001). It is important that in all these propositions no numerical values are used and thus algebraic relations are derived through geometry. It is also mentioned by Heath (2001) that Proclus (5th century AD) states that Eudemus of Rhodes claims that Pythagoreans discovered that when an equivalent parallelogram is fitted, then we have "simple parabola" of a region and not just equivalent region construction. When a parallelogram of smaller area than one square is fitted, we have a "parabola by lack of space" or "per ellipse", and when a parallelogram of larger area than one square is fitted, we have a "parabola by excess of space" or "per hyperbola". Later, Menaechmus used these names for plane curves resulting from conic sections. One of the properties of the parabolic curve is of particular interest and Archimedes calls it the "*equation of the parabola*" (Heath, Taliaferro & D'Ooge, 1952, p. 528. Reeder, 2015).

Proposition:
In a random parabolic arc and for two random points of this arc, the ratio of the squares of their lateral distances (ordinates) parallel to the chord of the arc from the top diameter of the arc is equal to the ratio of the respective traces' distances of the ordinates to the top diameter of the parabolic arc (abscissas).

An indirect statement also comes from Apollonius, who states in the introduction of his work that in his first four books he mainly deals with topics that have already been addressed by others before him (Bergsten, 2019). From the previous proposition, it follows:

$$\frac{(\Xi\Phi)^2}{(\Gamma\Theta)^2} = \frac{(\Phi\Sigma)}{(\Theta\Sigma)} \Rightarrow \frac{(\Gamma\Theta)^2}{(\Theta\Sigma)} = \frac{(\Xi\Phi)^2}{(\Phi\Sigma)} = c \Rightarrow (\Phi\Sigma)\cdot c = (\Xi\Phi)^2 \Rightarrow (\Phi\Sigma)\cdot c \cdot sin\alpha = (\Xi\Phi)^2 \cdot sin\alpha$$

where α is the angle between the top diameter of the arc and its chord, and it is interpreted that at any point of any arc of a parabolic curve, the local "Apollonian coordinates" of the point are (a) sides of a rhombus or a square (for the ordinate $\Xi\Phi$) and (b) sides of an equivalent rectangle or parallelogram (for the abscissa $\Phi\Sigma$ and the constant c) (figure 2). The constant c of the previous equation is proved to be equal to $\frac{latus\ rectum\ of\ parabola}{sin^2\alpha}$ (Ntontos, 2019, p. 73). Through this sentence, the concepts of "variable" and "co-variation of quantities", as well as the "oblique coordinate system" are introduced.

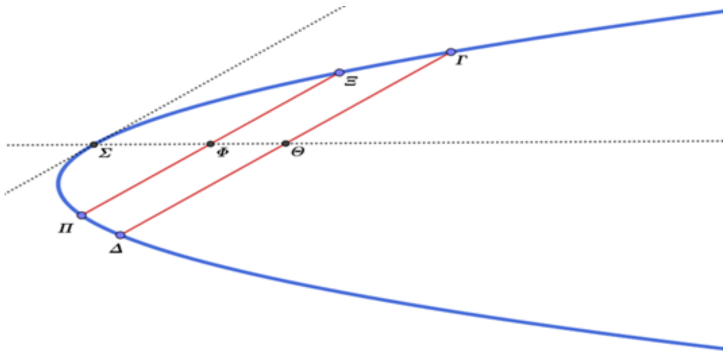

Figure 2: Parallel chords in a parabola

All these are embodied into this mathematical machine that will be analyzed and could also be considered as an area-measuring device. Its originality lies primarily in the proposed engineering design which deviates from geometric design or construction through dynamic geometry software, but also in S.T.E.M. activities for education.

2. Analysis of the Mechanism

2.1. The "parabola" or "fitting" of equivalent square

First will be described the "fitting" of an equivalent square on one side AB of a rectangular parallelogram $AB\Gamma\Delta$ and on one of its two adjacent right angles (let's assume at A) as described in proposition II.14 of "Elements" and according to Bergsten (2019). Let's also assume that the side AB of the rectangle is longer than side $B\Gamma$, and on the extension of BA a point Θ is defined so that $(A\Theta) = (A\Delta)$ (figure 3). Then, by constructing the midpoint M of $B\Theta$, a circle is drawn with center M and radius $(B\Theta)/2 = (M\Theta)$. By extending $A\Delta$, the circle is intersected at H, forming the right triangle $BH\Theta$. From the metric relations in right triangles (Proposition VI.8 of the "Elements"), it follows that segment AH is the mean proportional between segments AB and $A\Theta$, i.e., $(AH)^2 = (AB)\cdot(A\Theta) \Rightarrow (AH)^2 =$

$(AB) \cdot (A\Delta)$, and therefore the square $AEZH$ is equivalent with the rectangle $AB\Gamma\Delta$, ultimately resulting in the "squaring" of the rectangle and, by extension, the squaring of any rectilinear plane figure X of equal area.

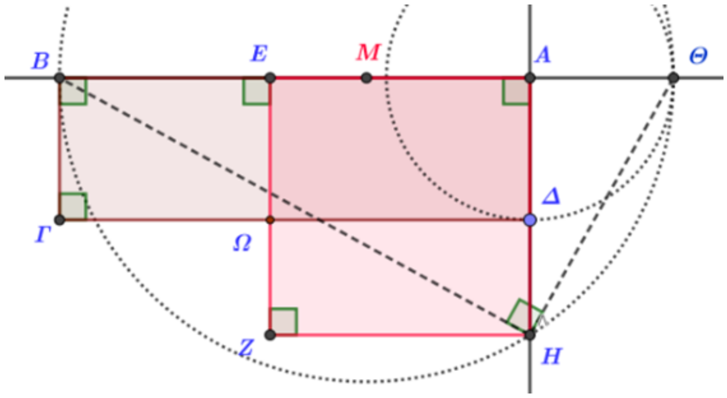

Figure 3: Geometric construction of squaring a rectangular parallelogram

When segment AB is fixed, this conclusion holds for any length of side $A\Delta$ ($A\Delta < AB$) of the rectangle. Also defining $(A\Delta) = \psi$ and $(AH) = \chi$, the previous relationship takes the form: $\chi^2 = c \cdot \psi \Leftrightarrow \chi^2/\psi = c = (AB)$, a relationship that reminds the inherent property that emerged from Menaechmus study of the parabola as a product of the perpendicular intersection of a right and a rectangular cone (orthotomus).

The point Ω has the following characteristics: (a) it is at a distance χ from the sliding axis AH and (b) it is at a distance ψ from the axis of the fixed "in length and position" side AB (figure 4). By continuously changing the length of the side $A\Delta$ of the rectangle $AB\Gamma\Delta$, the "simple fitting of a square" is continuously constructed, and the successive positions of point Ω show the changing side ($A\Delta = E\Omega$) of the rectangle and the changing side ($AH = \Omega\Delta$) of its equivalent square. By moving the cursor Δ a parabolic arc is drawn by the stylus to Ω, with one end being the vertex A of the corresponding parabola.

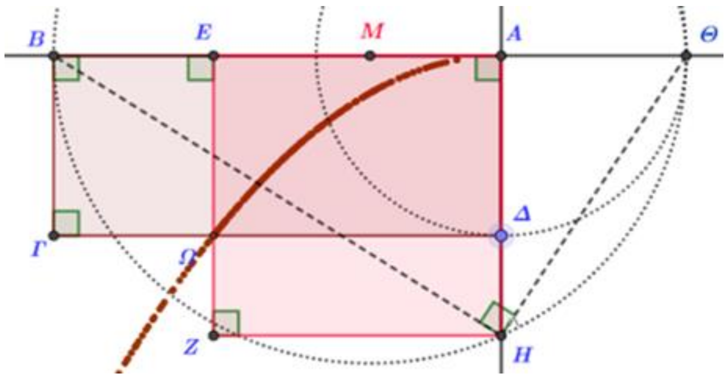

Figure 4: Kinematic drawing of a parabola by squaring a rectangular parallelogram
https://www.geogebra.org/m/gnasrsxe

In the case where AB is shorter than $B\Gamma$, Sardelis and Valahas (2017) suggest that by keeping fixed the side AB of the rectangle $AB\Gamma\Delta$, a square equal in area to $AB\Gamma\Delta$ is constructed, and the same relationship for the varying length of the side $A\Delta$ and the corresponding side of the square $AEZH$ emerges, i.e., $(AH)^2 = (AB) \cdot (A\Theta) \Leftrightarrow (AH)^2 = (AB) \cdot (A\Delta)$ (figure 5). The square $AEZH$ is always equivalent to the rectangle $AB\Gamma\Delta$, thus geometrically constructing the "squaring" of the rectangle and, by extension, of any equivalent rectilinear shape X. By also defining $(A\Delta) = (E\Omega) = \psi$ and $(AH) = (\Omega\Delta) = \chi$, the previous relationship becomes $\chi^2 = c \cdot \psi \Leftrightarrow \chi^2/\psi = c = (AB)$. Placing a stylus at point Ω and moving point Δ, a parabolic arc is produced, with one end being the vertex A of the corresponding parabola.

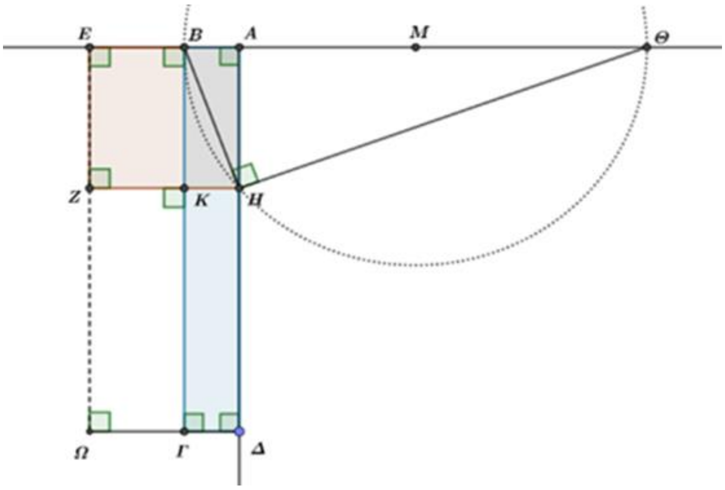

Figure 5: Geometric construction of squaring a rectangular parallelogram

Something similar resulted from the perpendicular intersection of a right and a rectangular cone (orthotomus) (figure 6). It was proven that the "ordinate" ΓM (according to Apollonius) of point Γ of the curve is the mean proportional between the "abscissa" MK and the fixed distance ΣK on the generatrix of the lateral conical surface, having constructed a continuous squaring of a rectangle whose one side is the constant segment $2 \cdot \Sigma K$= "latus rectum" and the other side is the abscissa MK. In other words, $M\Gamma^2 = 2 \cdot \Sigma K \cdot MK$ (Ntontos, 2019, p. 35).

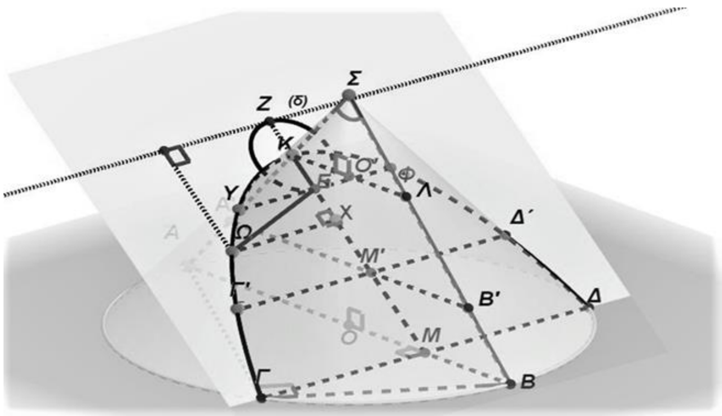

Figure 6: Orthotomus of Menaechmus

2.2. Vertex of parabola

In the dynamic environment of GeoGebra software, the simulation of the machine which can move continuously the point Δ (adhering to the geometric specifications mentioned) and, with a stylus placed at point Ω ("trace on"), a curve is drawn. This curve, for every change in the side $A\Delta$ of the rectangle $AB\Gamma\Delta$, defines the side of the corresponding equivalent square that "fits" on side AB and at angle A. In the metric relationship $\chi^2 = c \cdot \psi \Leftrightarrow \chi^2/\psi = c = (AB)$, if we search for the positions of point Ω when $\chi = \psi$, it follows that $\chi = \psi \Rightarrow \chi^2 = c \cdot \chi \Rightarrow \chi \cdot (\chi - c) = 0 \Rightarrow \chi = \psi = 0$, or, $\chi = \psi = c = (AB)$, which is also verified through the software (figure 7).

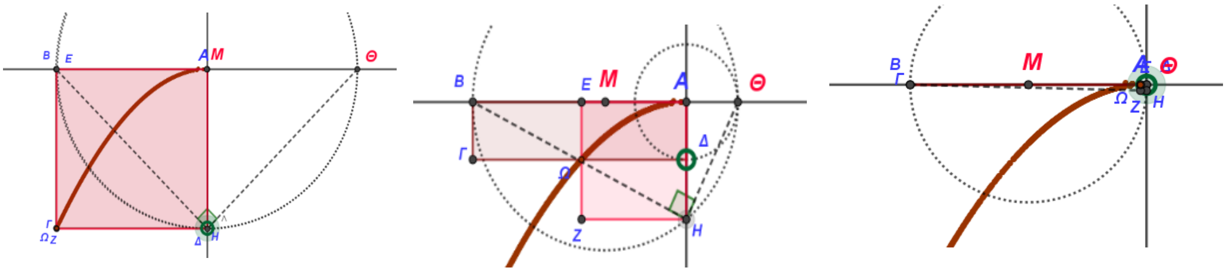

Figure 7: Successive positions of the cursor Δ from χ=0 to χ=ψ,
https://www.geogebra.org/m/gnasrsxe

Because the square has been "fitted" on the side AB and angle A, this is a strong indication that at point A the curve is smooth, having the tangent line AB. By constructing the symmetric of the drawn parabolic arc, with axis of symmetry being the line AH, this construction can be extended so that the line AH serves as the vertex diameter at point A of the parabolic arc, which finally is the axis of symmetry of the corresponding parabola, because the line $A\Delta$ is by construction the perpendicular bisector of the chord $\Omega'\Omega$ for any position of the cursor Δ (figure 8).

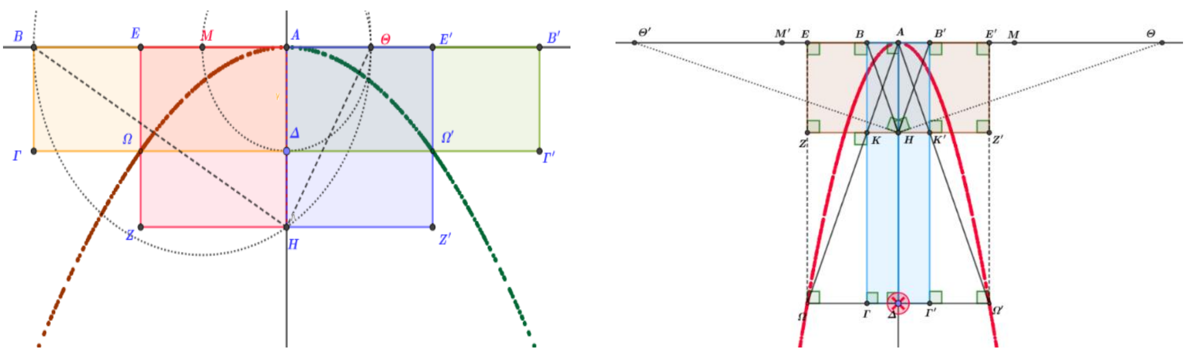

Figure 8: Construction of symmetrical parabolic arc
https://www.geogebra.org/m/ywfuh4uc

2.3. Collinearity of three points

In the case where the parameter $AB > B\Gamma$, the diagonal $A\Omega$ of the rectangle $AE\Omega\Delta$ (figure 9) passes through the point K, which is the imaginary intersection point of HZ and $B\Gamma$. This is because if it did not pass through K, then AK would intersect EZ at another point

Ω' with corresponding distances χ and ψ'. By subsequently bringing a segment $\Gamma\Delta$ from Ω' parallel to the sides AB and KH, the rectangle $AB\Gamma\Delta$ is equal in area to the square $AEZH$ due to the property of the diagonal of a parallelogram creating equal and equal-area triangles (Proposition I.43 of the "Elements"), and it follows that $c \cdot \psi' = \chi^2$. However, for Ω, the relationship $c \cdot \psi = \chi^2$ also holds, and therefore $\psi = \psi'$, meaning the points Ω and Ω' coincide, and the diagonal AK of the parallelogram $ABKH$ also passes through the fourth vertex Ω of the rectangle $EA\Delta\Omega$, which is a point on the parabola.

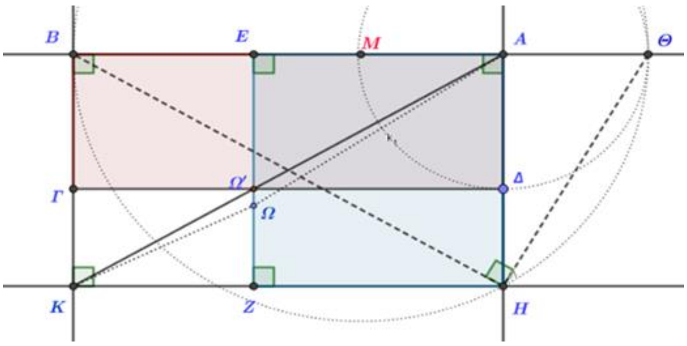

Figure 9: Collinearity of points A, Ω and K when $AB > B\Gamma$

In the same way, it is proven that in the case where the parameter $AB < B\Gamma$, the diagonal AK of the rectangle with sides AB (the fixed side of the rectangle) and AH (the side of the square), passes through the vertex Ω of the rectangle $AE\Omega\Delta$, whose distances from the vertical side of the square $(\Omega\Delta) = \chi$ and from the horizontal side of the rectangle $(\Omega E) = \psi$, connected by the relation $c \cdot \psi = \chi^2$ (figure 10).

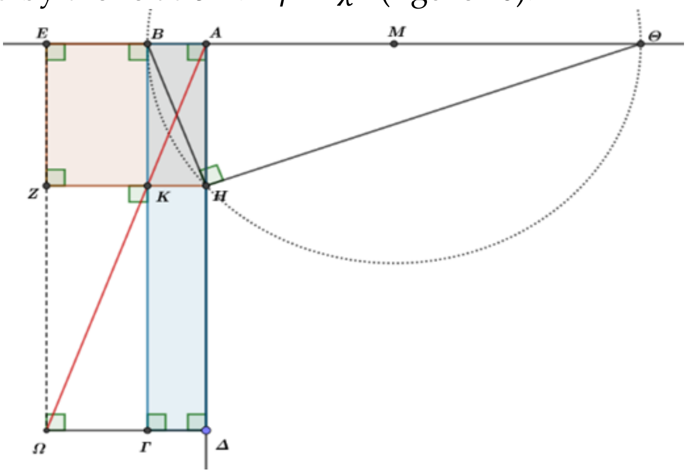

Figure 10: Collinearity of points A, Ω and K when $AB < B\Gamma$

Finally, a necessary and sufficient condition for the quadrature of the rectangle and for the tracing of a parabolic curve by the stylus at Ω is the collinearity of the points A, Ω, K.

3. Synthesis of the Mechanism

3.1. The simple mechanism

The rod AB is hinged perpendicularly to rods $A\Delta$ and $B\Gamma$ (where $A\Delta = B\Gamma$) (figure 11). Points E and H on AB and $A\Delta$ respectively, are equidistant "appropriately" from A and point H by the placement at H of a 45° joint with one degree of freedom allowing only sliding on rod $A\Delta$ (prismatic joint). The rod EZ is also articulated at an angle of 90° and prismatic on AB so that it can slide while remaining perpendicular. At points K, Z and H the rod KZH is articulated perpendicularly and prismatically at points H, Z and K.

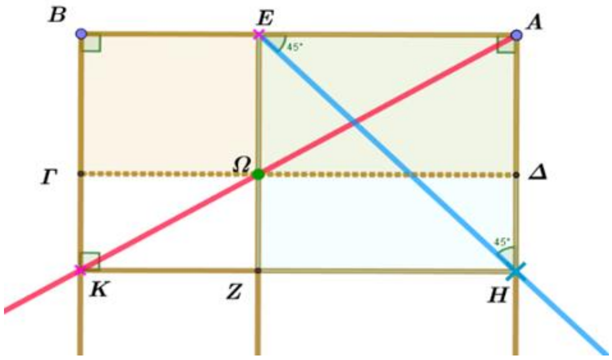

Figure 11: Squaring a rectangular parallelogram

At the intersection point Ω of the side EZ of the square $AEZH$ and the diagonal AK of the rectangle $BAHK$, a stylus Ω is articulated rotatably and double-prismatically (figure 12). The side AH of the square $AEZH$ acts as an independent variable, as point A is fixed and point H (mechanism input) slides along the rod $A\Delta H$, and its movement activates the mechanism, moving the rods $E\Omega Z$, HZK parallel to their initial positions and drawing a curve with the stylus Ω.

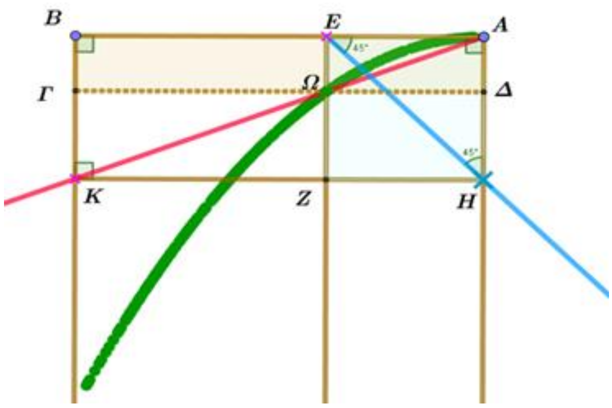

Figure 12: Kinematic drawing of a semi-parabola by squaring a rectangular parallelogram https://www.geogebra.org/m/uhdzghyu

In each position of the cursor H, the changing (in terms of its dimensions) square $AEZH$ has the same area (equivalent) with the variable rectangle $AB\Gamma\Delta$ (with a fixed side AB and an imaginary side $\Gamma\Omega\Delta$), due to the diagonal $A\Omega K$ in the rectangle $BAHK$. In other words, "the square of the changing (AH) is equal to the product of the changing ($A\Delta$) and the constant (AB)".

3.2. The complex "upright" mechanism

This mechanism includes the symmetrical part of the simple mechanism with respect to the sliding rod $A\Delta H$ (figure 13). Specifically, in the extension of BA by an equal length AB', a rod is articulated perpendicularly at B' with the ability to slide along the rod BAB'. On the cursor H, a vertical rod HE' is articulated with HE, with a prismatic joint also at H and on this rod. Due to 'articulated geometry,' the isosceles and right triangle HAE' is formed, which is equal to HAE for any position of H.

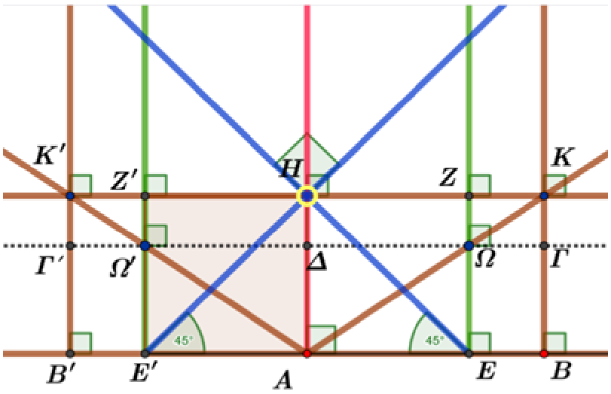

Figure 13: The complex "upright" mechanism

The extension of KH is articulated perpendicularly at K' of $B'\Gamma'$, and the perpendicular rod of BB' at E' (prismatically on BAB') is articulated perpendicularly at Z' of KK' (and prismatically on $KZHZ'K'$). Finally, at the fixed-point A, the diagonal of the rectangle $AHK'B'$ is rotatably articulated, which is also articulated rotatably and double-prismatically at point Ω' of the side $E'Z'$ of the square $AHZ'E$, where a stylus is also placed. By moving the cursor H, the parabolic arc ZAZ' is drawn (figure 6). For any position of the cursor H, the angle AHE is 45°, due to the articulation of 45° and the quadrilateral $AHZE$ is a square. Due to symmetrical construction of this mechanism, the rods HE and HE' are always perpendicular to each other due to the articulations, and the quadrilateral $AE'Z'H$ is also a square equal to $AEZH$ and symmetrical with respect to the rod AH. Due to the axial symmetry of the mechanism with respect to the rod AH, the symmetry of the curves drawn by the two styluses is obvious (figure 14). Additionally, by continuously reducing the distance AH, it becomes apparent through the machine that at point A, the curve (although it will be difficult for the two points to meet constructively) is 'smooth' due to the 'fitting' of the two squares at A, having collinear sides AE and AE' and successive right angles at A, making A the vertex of the Parabola and the line AH its axis of symmetry.

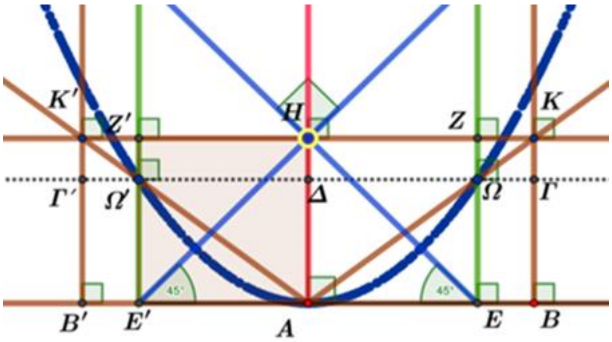

Figure 14: Kinematic drawing of a parabola by squaring two symmetric rectangular parallelograms, https://www.geogebra.org/m/b7vque34

3.3. The complex "oblique" mechanism

The joints at points $B, E, A, E', B', \Omega, \Omega', K, Z, H, Z', K'$ are rotational, and some of them are prismatic or double-prismatic. The mechanism can rotate by an angle α without sliding the cursor H and thus without any movement of its rods (figure 15). The perpendicularity at H is maintained due to the orthogonal articulation, but in order to maintain the parallelism of the rods during the sliding of the cursor H, and consequently the existence of the parallelograms defined within the mechanism, the angles at points $B, E, A, E', B', K, Z, H, Z', K'$ must remain equal to α. This can be achieved either through appropriate joints that eliminate their rotational movement at these points or by placing metallic circular sectors of angle α at these points.

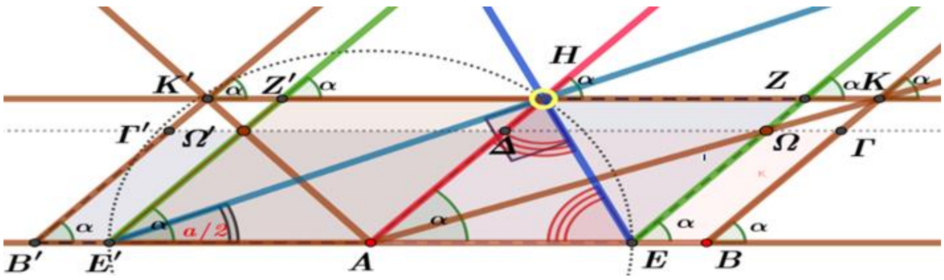

Figure 15: The complex "oblique" mechanism, https://www.geogebra.org/m/yedcwtkb

Additionally, to maintain the isosceles nature of the triangles AHE and AHE' during the sliding of the cursor H, at E' either through an appropriate joint that eliminates its rotational movement or by placing a metallic circular sector of angle $\alpha/2$. Then the triangles AHE and AHE' will remain isosceles with vertex A and the quadrilaterals $AEZH$ and $AE'Z'H$ will remain rhombuses that are equal to each other. Moving the cursor H, the styluses Ω and $\Omega'$are drawing a parabolic arc.

3.4. Degrees of freedom and symmetries

The variable quantities of the machine are the quantities χ and ψ as well as the angles defined by the rods AK and AK' with their articulated rods $A, \Omega, K, \Omega', K'$. It is easily shown that all these angles are functions of χ and ψ. The quantities χ and ψ are connected by the relation $\chi^2 = c \cdot \psi$ and it follows that the simple and complex upright parabolographs have one degree of freedom. From this relation, it follows that $\chi^2 = c \cdot \psi \Leftrightarrow \sqrt{\chi^2} =$

$\sqrt{c \cdot \psi} \Leftrightarrow \chi = \pm\sqrt{c \cdot \psi}$, which indicates the complex upright machine traces a symmetrical parabolic arc with the axis of symmetry being the sliding line of the cursor H.

3.5. Machine's parameters

Increasing the length of side AB, the parabolograph within the frame of its dimensions "flattens" the traced parabola, while if we decrease this length, the parabolograph "curves" (figure 16). This can be demonstrated by the relation $\chi^2 = c \cdot \psi$, which connects the variables of the machine. For a fixed value of the variable $\chi = (\Omega\Delta) = (AH)$, when the value of the parameter $c = (AB)$ increases, the value of the variable $y = (\Omega E)$ must decrease, "flattening" the traced curve (and vice versa).

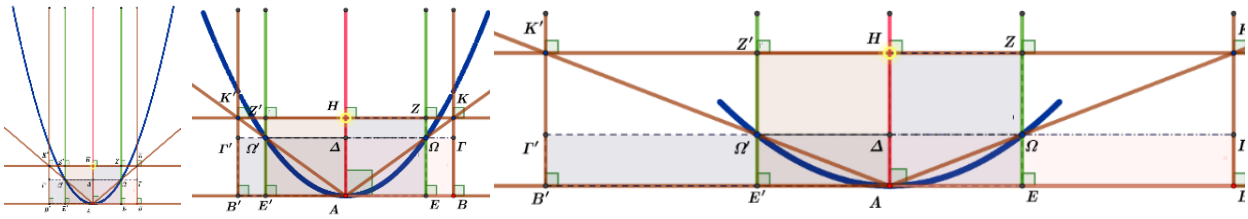

Figure 16: Curvature parameter of the traced parabolic arc https://www.geogebra.org/m/uneq83mb

For every parabola drawn by the machine when the angle $\alpha = 90^o$, we can find its focus and directrix, given that we know the "latus rectum" $c = (AB)$ and the axis of symmetry AH of the parabola. According to Bergsten (2019), the method of "applying areas" works equally well in the non-right-angle case. The following figure depicts the construction of the parabola according to Archimedes' proposition. Rotating the sliding rod of the cursor H around the fixed-point A, all the parallel rods of the mechanism rotate by the same angle, transforming the rectangle $AB\Gamma\Delta$ into a parallelogram $AB\Gamma\Delta$ and the square $AEZH$ into a rhombus $AEZH$, which have a common angle α at A and collinear sides AB and AE. The parallelogram $AB\Gamma\Delta$ is equivalent to the rhombus $AEZH$ according to Proposition I.43 of "Elements" and results in: $(AEZH) = (AE) \cdot (H\Pi) = (AE) \cdot (AH) \cdot sina = (AE) \cdot (\Omega\Delta) \cdot sina = \chi \cdot \chi \cdot sina = \chi^2 \cdot sina$. Due to the equality of the areas: $(AEZH) = (AB\Gamma\Delta) \Rightarrow \chi^2 \cdot sina = (AB) \cdot (A\Delta) \cdot sina \Rightarrow \chi^2 = c \cdot \psi$ which means that the point Ω has oblique distances $\chi = (\Omega\Delta) = (AH)\ and\ \psi = (\Omega E) = (A\Delta)$ from $A\Delta$ and AB respectively, such that: $\chi^2 = c \cdot \psi \Leftrightarrow \frac{\chi^2}{\psi} = c = (AB)$ (figure 17).

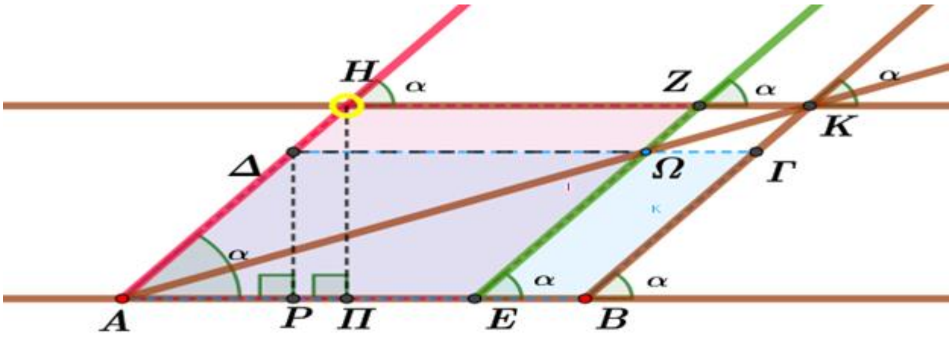

Figure 17: The simple oblique parabolograph, changing the angle α.

Likewise, for the corresponding Ω', the same algebraic relationship holds for its oblique distances from $A\Delta$ and AB'. Due to the symmetry with respect to $A\Delta$ (when $\alpha = 90^o$), cursor H remains the midpoint Δ of the chord $\Omega\Omega'$, even after the change in the measure of angle α, as there was no corresponding change in lengths of the rods (figure 18).

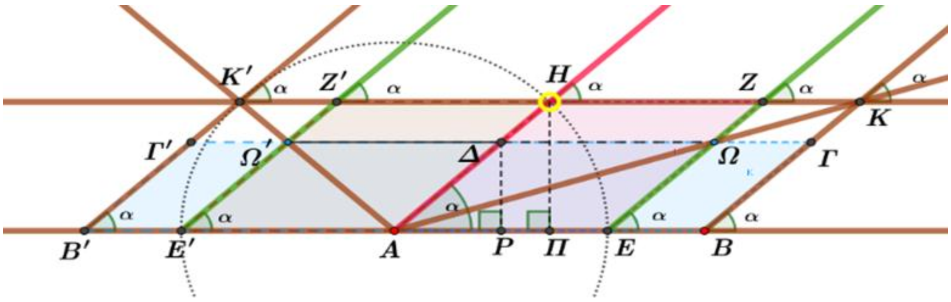

Figure 18: The complex oblique parabolograph, changing the angle α.

This shows that the successive positions of the points Ω and Ω' create a parabolic arc with the vertex at point A and its principal diameter being $A\Delta H$, without point A of the arc being the vertex of the corresponding parabola.

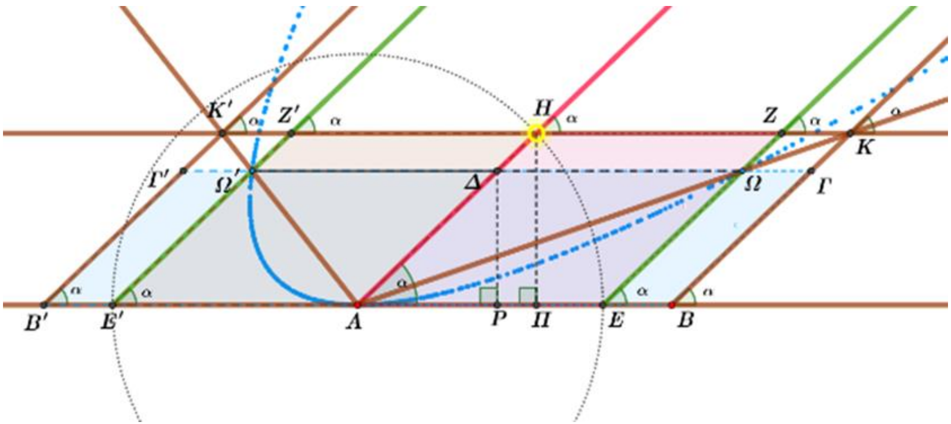

Figure 19: The parabolic arc that results after α value of the angle α,
https://www.geogebra.org/m/yedcwtkb

The angle α is second parameter of the machine that causes changes in the local and general curvature of the parabola (figure 20).

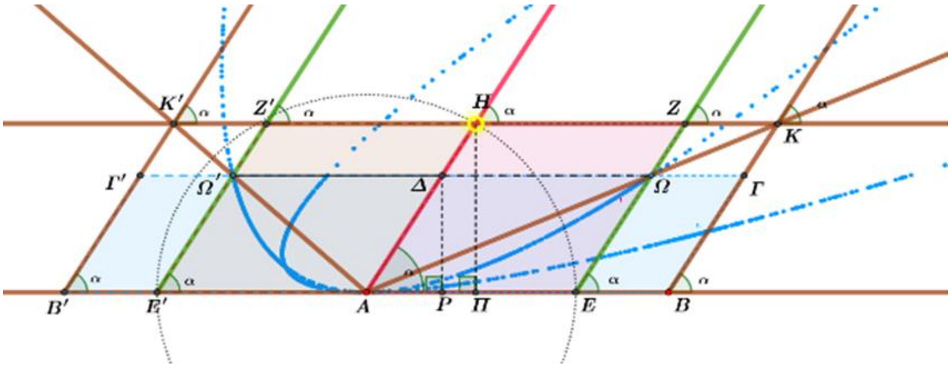

Figure 20: The parabolic arc that results after two values of the angle α,
https://www.geogebra.org/m/gu7dtygb

This curve (according to the proposal used by Archimedes) is a parabolic arc, because for all its points, the ratio of the squares of the "ordinates" to the corresponding "abscissas" equals the parameter $(AB) = c$ of the mechanism. It has also been shown that the algebraic relation (Ntontos, 2019, p. 71):
$\frac{\chi^2}{\psi} = \frac{latus\ rectum\ of\ the\ drawn\ parabola}{sin^2\alpha} \Rightarrow \chi^2 = \frac{latus\ rectum\ of\ the\ drawn\ parabola}{sin^2\alpha} \cdot \psi$, and using the machine it follows that $\chi^2 = (AB) \cdot \psi$. From these relations, it follows that $(AB) = \frac{latus\ rectum\ of\ the\ drawn\ parabola}{sin^2\alpha} \Rightarrow$ "$latus\ rectum\ of\ the\ drawn\ parabola$" $= (AB) \cdot sin^2\alpha$.
For each value of the angle α, between the sliding line $A\Delta$ and the fixed side AB of the parallelogram, this mechanism draws a parabolic arc that has *latus rectum* equal to $(AB) \cdot sin^2\alpha \leq (AB)$. Therefore, when the angle α decreases, the latus rectum of the drawn parabola curves (and vice versa).

It is a property of the parabola that each of its points Σ constitutes the vertex of all its arcs with chords parallel to the tangent line at that point. When the angle at A in the mechanism is $\alpha = 90^o$, for every point of the traced arc that have vertex Σ, which has oblique distances χ and ψ from the corresponding principal diameter and from the tangent at the vertex Σ of this arc, the relation $\frac{\chi^2}{\psi} = \frac{latus\ rectum\ of\ the\ "upright"\ parabola}{sin^2\beta} = \frac{(AB)}{sin^2\beta} \geq (AB)$= latus rectum of the "upright" parabola is valid, where β is the angle between chord and axis of symmetry of this parabola. We could say that at every such point Σ of this parabola with $\alpha = 90^o$, there corresponds a hypothetical parabola with a greater "latus rectum" which increases as this point moves away from the vertex of the initial "upright" parabola because the angle β decreases at the same time.

3.6. Focus and Directrix

In the parabolic arc, drawn by the "upright" mechanism, finding the focus and directrix is a trivial case, since the "latus rectum" equals the length of the parameter $c = (AB)$. For the remaining cases the corresponding parabolic arc is first drawn. The direction of the rod $A\Delta H$ is the diameter of the drawn parabolic arc and parallel to its axis of symmetry of the parabola, because Δ is the midpoint of $\Omega\Omega'$ for any position of cursor H as it moves along the rod $A\Delta H$. Therefore, parallel chords $\Omega\Omega'$ are created, whose respective midpoints Δ must be collinear[ii]. By selecting a random point T on the arc and drawing from it a perpendicular line to the diameter $A\Delta H$, this line is also perpendicular to the axis of symmetry of the parabola and intersects the arc at point Σ (figure 21). The midpoint M of the chord $T\Sigma$ belongs to the axis of symmetry of the parabola, which we draw by making the perpendicular to the chord $T\Sigma$ at midpoint M. Additionally, the intersection point O of the axis of symmetry with the arc is also the vertex of the parabola.

The directrix is drawn as a tangent line to the circle $(K, \frac{(AB)\cdot sin^2\alpha}{4})$ parallel to the chord $T\Sigma$, as it has been proven that by turning the mechanism by an angle α, the parabolic arc that is drawn has "*latus rectum*" ($o\rho\theta i\alpha$) equal to $(AB) \cdot sin^2\alpha$.

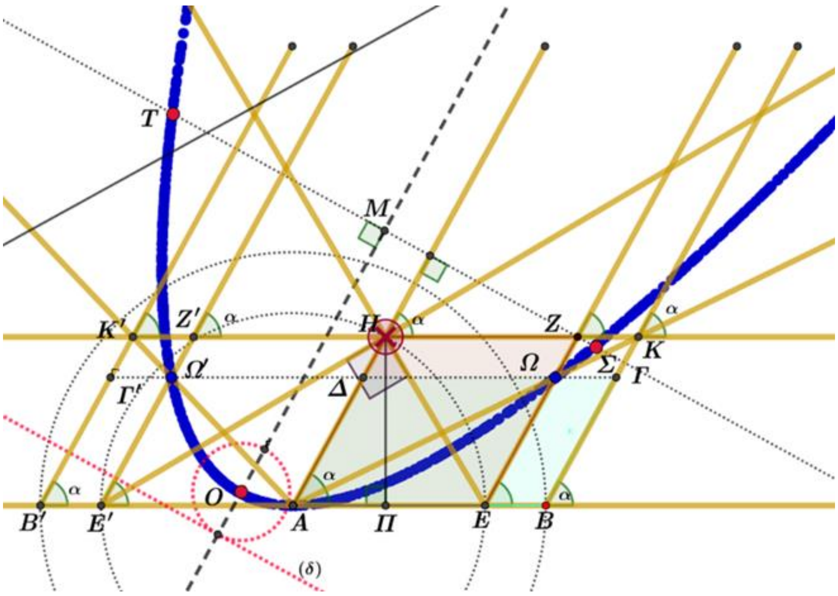

Figure 21: Construction of axis of symmetry, directix and focus on a parabolic arc drawn by the oblique parabolograph

To be constructed geometrically a segment with length equal to $(AB) \cdot sin^2\alpha$, it is sufficient for the cursor H to coincide with the "moving" Δ at the position where $AB = AE$ and for the rhombus $AHZE$ to be matched with the parallelogram $A\Delta\Gamma B$, and $\chi = \psi$ for that specific position. In this state, the perpendicular distance $(\Delta P) = (H\Pi)$ of point Δ from AB is equal to $(A\Delta) \cdot sin\alpha = (AB) \cdot sin\alpha = c \cdot sin\alpha$ (figure 22).

[ii] The continuous collinearity of the points Δ, Ω, Ω', as well as the continuous bisection of Ω' in Δ, for any position of the cursor H on AD, are mechanically ensured.

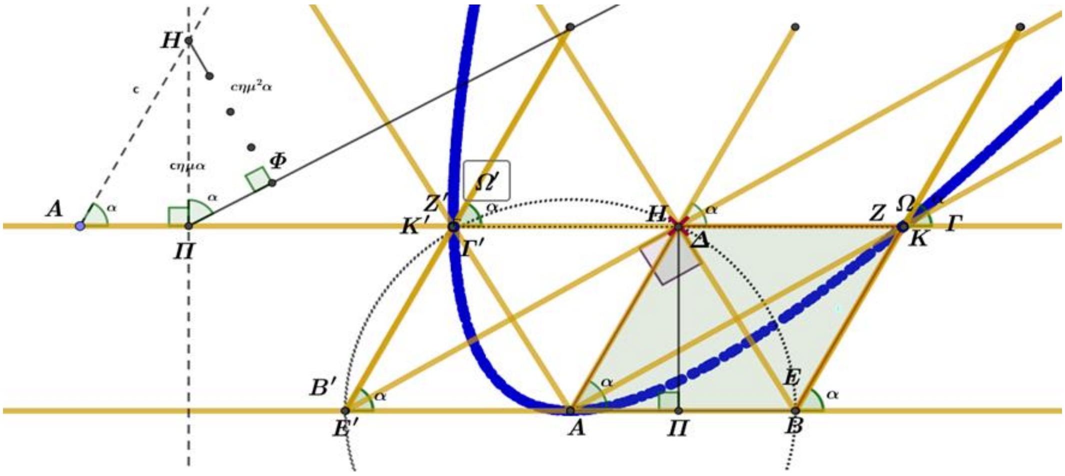

Figure 22: Geometric construction of a segment with length equal to $(AB) \cdot sin^2\alpha$

Then, the segment of length $(AB) \cdot sin^2\alpha = (H\Phi)$ can be constructed through the right triangle $H\Pi\Phi$ ($\widehat{\Phi} = 90^o$ $\kappa\alpha\iota$ $\widehat{\Pi} = \alpha$). Dividing the segment $\Delta\Phi$ into four equal parts of length $\frac{\boldsymbol{c \cdot sin^2\alpha}}{\boldsymbol{4}}$ (figure 22) the circle $(O, \frac{c \cdot sin^2\alpha}{4})$ constructs the focus F and the directrix δ of the parabola drawn by the mechanism (figure 22).

4. Conclusions

In this article, a new mathematical machine was presented which "squares" or "rhombuses" rectangles or parallelograms respectively, while simultaneously drawing parabolic arcs. The idea originated from Sardelis, D., and Valahas, T. (2012), as well as Bergsten (2015), based on the Pythagorean theory of "simple parabola- fitting of region" and not just on the construction of equal-area regions as described in Euclid's "Elements". The originality of this work lies in the fact that this mechanism can now be designed and constructed, highlighting the unique mechanical presence of the joint-cursor H which is not showcased in the simulation via dynamic geometry software. Specifically, in the mechanical construction of the simple mechanism, an additional rod is articulated at H with an angle of 45° and slides along the rod $A\Delta$ (figure 23). Its other end is prismatically articulated at point E of AB. This constantly creates a right and isosceles triangle AEH for any position of the cursor H, which in the simulation does not need to exist, but only a command to construct a circle with center A and radius AH (with H moving along $A\Delta$). In this way, the desired equal-area square appears within the mechanism.

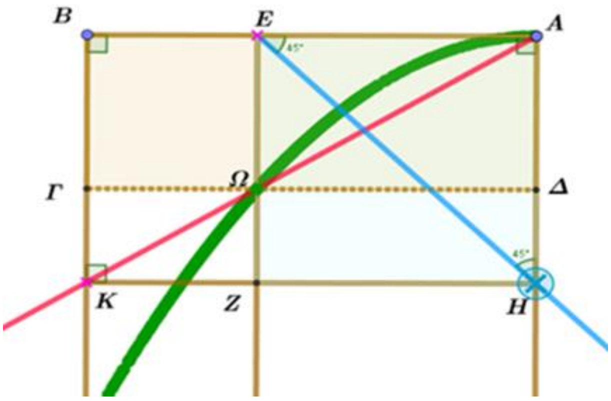

Figure 23: The simple "upright" mechanism

By placing the symmetrical part of the mechanism, an angle of $45^o + 45^o = 90^o$ is summed at H, which is a prerequisite for the mechanism to have two right, equal, and isosceles triangles AHE and $AH'E$ which through the articulated parallels create the two equal squares that are also equal in area to the rectangles with side $(AB) = (AB') = c$ (figure 24). In contrast, in the software, only a single circle-drawing command is required. Therefore, the bars EH and $E'H$ must be perpendicularly hinged to each other at H that slides on rod $A\Delta H$.

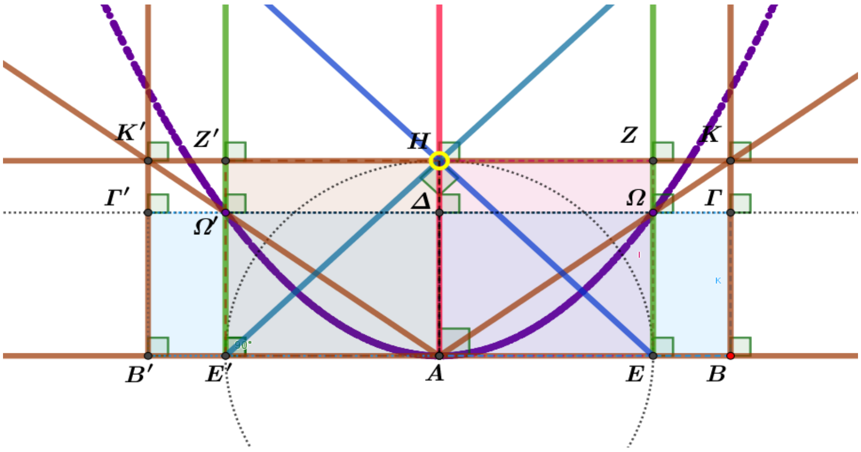

Figure 24: The complex "upright" mechanism

In the case of the complex-oblique mechanism resulting from rotating the entire upright complex mechanism by an acute angle α at all joints B, E, A, E', B' and K, Z, H, Z', K' (without sliding their prismatic joints), additional interventions are required, which the software "hides" through commands related to parameters or the drawing of parallel lines (figure 25). Specifically, after rotating the mechanism, the rotational joints need to be mechanically locked, placing circular sectors of angle α. The prismatic joints can then operate as the H cursor moves.

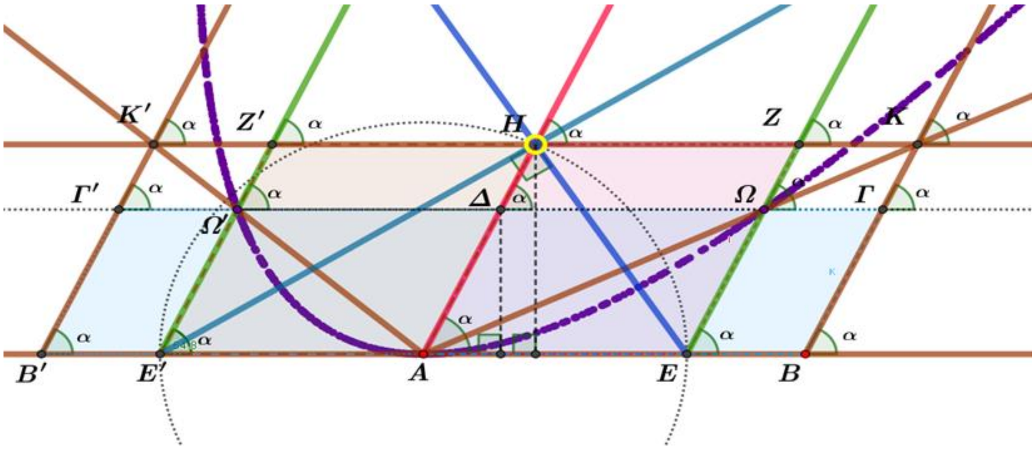

Figure 25: The complex "oblique" mechanism

This entire process of analysis and mechanical design of the mechanism "under construction", creates a challenging environment for students and beyond. It would certainly be better to have Pythagorean machine in front of us, to describe and analyze its structure, but at this stage of mechanical design, things are more difficult yet also more intrinsic. The process is both analytical and synthetic, and particularly S.T.E.M.. At all stages, paths are made through Geometry, Kinematic Geometry, Engineering, Software, and the Teaching of all these.

Conflict of Interest Statement

The authors declare no conflicts of interest.

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